

# Extended iterative scheme for QCD: three-point vertices

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**Abstract.** In the framework of a generalized iterative scheme introduced previously to account for the non-analytic coupling dependence associated with the renormalization-group invariant mass scale  $\Lambda$ , we establish the self-consistency equations of the extended Feynman rules ( $\Lambda$ -modified vertices of zeroth perturbative order) for the three-gluon vertex, the two ghost vertices, and the two vertices of massless quarks. Calculations are performed to one-loop-order, in Landau gauge, and at the lowest approximation level ( $r = 1$ ) of interest for QCD. We discuss the phenomenon of compensating poles inherent in these equations, by which the formalism automatically cancels unphysical poles on internal lines, and the role of composite-operator information in the form of equation-of-motion condensate conditions. The observed near decoupling of the four-gluon conditions permits a solution to the 2-and-3-point conditions within an effective one-parameter freedom. There exists a parameter range in which one solution has all vertex coefficients real, as required for a physical solution, and a narrower range in which the transverse-gluon and massless-quark propagators both exhibit complex-conjugate pole pairs.

## 1 Summary of the extended iterative scheme

The spontaneous emergence of a renormalization-group (RG) invariant mass scale  $\Lambda$  [1] from the renormalization process is arguably the most important nonperturbative effect in strictly renormalizable quantum field theories, since this quantity sets the scale for all dimensionful observables (except those dominated by heavy extraneous masses). Its coupling dependence,

$$\begin{aligned}
 (\Lambda^2)_R &= \nu^2 \exp \left\{ -2 \int^{\nu} \frac{dg'}{[\beta(g')]_R} \right\} \\
 &= \nu^2 \exp \left\{ -\frac{(4\pi)^2}{\beta_0 g(\nu)^2} [1 + \mathcal{O}(g^2)]_R \right\}, \quad (1.1)
 \end{aligned}$$

(where  $R$  denotes a renormalization scheme and  $\nu$  the arbitrary renormalization scale within  $R$ , and where  $\beta_0 > 0$  in an asymptotically free theory) is non-analytic in a way that will always remain invisible in a perturbation expansion around  $g^2 = 0$ . Moreover, the several known obstructions [2] to the existence and uniqueness of a Borel transform of the perturbation series all have to do, at least qualitatively, with the presence of this scale. There exists, therefore, the intriguing *possibility that accounting systematically for the  $\Lambda$  dependence of correlation functions may be the minimal step beyond perturbation theory needed to define a strictly renormalizable theory uniquely.* (Here “ $\Lambda$  dependence” does not, of course, refer to a mere

reparametrization of the perturbation series, as obtained by solving (1.1) for  $g$  in terms of  $\Lambda$ , or equivalently by leading-logarithms resummation. Genuinely nonperturbative  $\Lambda$  dependence, exemplified by the way vacuum condensates occur in operator-product expansions (OPE), is typically polynomial or inverse-polynomial).

The present paper elaborates on a specific scheme, outlined earlier in this journal [3], of accounting systematically for the  $\Lambda$  dependence, under the restriction (which in an asymptotically free theory turns out to be a weak one) that the known standard technique of renormalization remain applicable with at most inessential modifications. This scheme takes the form of an *extended iterative solution* to the integral equations for correlation or vertex functions, starting from a set of extended Feynman rules for the superficially divergent basic vertices of the theory, as distinct from the ordinary Feynman rules (bare vertices)  $\Gamma^{(0)pert}$ , whose iteration generates the perturbative series. These extended vertices are quantities of zeroth order ( $p = 0$ ) with respect to the perturbative  $g^2$  dependence but contain a “seed dependence” on the nonanalytic scale (1.1), approximated systematically. Under the combined requirements of globality (in order to be applicable in loop integrals, the approximation must in principle be valid over the entire momentum range) and of the preservation of power counting (as a basic prerequisite of standard renormalization technique), the choice of approximating functions is remarkably unique: they must be functions *rational* with respect to  $\Lambda$  and therefore (since  $\Lambda$  is di-

mensionful) also with respect to momentum variables. For the representation of the ( $p = 0$ ) nonperturbative  $\Lambda$  dependence, rational approximants perform the same basic role that polynomial approximants play for the perturbative  $g^2$  dependence. The poles of these rational functions, together with the numerator zeroes, will in general form discrete approximations to branch cuts of the vertices in their complex-momentum planes. We follow [3] in denoting the extended rules by  $\Gamma^{[r,0]}$ , with  $r$  the (denominator) degree of rational approximation. At each level  $r$ , the  $\Gamma^{[r,0]}$  approach the  $\Gamma^{(0)pert}$  in the ‘‘perturbative limit’’  $\Lambda \rightarrow 0$ ; on the other hand their sequence as  $r$  increases may be viewed as an analytic continuation-through-resummation of the zeroth-order terms of the OPE.

The central problem of such a method is to demonstrate dynamical self-consistency of these extended Feynman rules. The integral equations (Dyson-Schwinger, or DS, equations) for the set  $\Gamma$  of proper vertex functions  $\Gamma_N$ , where  $N$  stands summarily for the number and types of external legs, take the schematic form

$$\Gamma_N = \Gamma_N^{(0)pert} + \left(\frac{g_0}{4\pi}\right)^2 \Phi_N[\Gamma^{(0)pert}, \Gamma], \quad (1.2)$$

where  $g_0$  is the bare gauge coupling, and  $\Phi_N$  a set of nonlinear dressing functionals, containing loop integrals over combinations of  $\Gamma$ 's. With the inhomogeneous terms always given by the bare  $\Gamma^{(0)pert}$ , iterating instead around a nonperturbatively modified set of starting functions  $\Gamma^{[r,0]}$  can work only if the interaction terms  $\Phi_N$ , at each order  $l$  of the iteration, not only generate an  $l$ -th order, ‘‘quasi-perturbative’’ power correction in  $g^2(\nu)$ , but also reproduce the nonperturbative parts,  $\Gamma^{[r,0]} - \Gamma^{(0)pert}$ , of the new zeroth-order input: as compared to perturbation theory, the solution must be able to ‘‘establish its own zeroth order’’. To have the functionals  $\Phi_N$ , in spite of their  $g_0^2$  prefactor, produce certain terms of *zeroth* order in  $g^2$  is not trivial. Moreover, for the method to be practical, the number of extended Feynman rules should remain finite (and small), as in perturbation theory, and in view of the notorious infinite hierarchical coupling in eqs. (1.2) – with each  $\Phi_N$  coupling to  $\Gamma$ 's up to  $\Gamma_{N+1}$  or even  $\Gamma_{N+2}$  –, this is again nontrivial.

The mechanism discussed in [3] for simultaneously ensuring both objectives exploits the structure (1.1) of the scale  $\Lambda$  in conjunction with the renormalizable divergence structure of the theory. To briefly describe its main line, let

$$\{c\}^{[r]} = \{c_{r,1}, c_{r,2}, \dots, c_{r,k_r}\} \quad (1.3)$$

be the complete set of  $k_r$  numerator coefficients of the level- $r$  rational approximants  $\Gamma_N^{[r,0]}$  for all superficially divergent vertices (of which, we recall, there are seven in covariantly quantized QCD), and let

$$\{d\}^{[r]} = \{d_{r,1}, d_{r,2}, \dots, d_{r,m_r}\} \quad (1.4)$$

be the complete set of  $m_r$  denominator zeroes (pole positions) in units of  $\Lambda^2$ . Both  $c$ 's and  $d$ 's are dimensionless,

real numbers. Then

$$\Gamma_N^{[r,0]} = \Gamma_N^{(0)pert} + \Delta_N^{[r]}(\{c\}_N^{[r]}, \{d\}_N^{[r]}; \Lambda), \quad (1.5)$$

in a notation suppressing momenta and all other variables not immediately pertinent to the argument. Here  $\{c\}_N^{[r]}$  denotes the subset of  $\{c\}^{[r]}$  appearing in the vertex  $\Gamma_N$ , etc. Upon evaluating, say, the first iteration (one-loop order,  $l = 1$ ) of (1.2), dimensionally regularized in  $D = 4 - 2\epsilon$ , with the functions (1.5) as input, one obtains after some algebraic decomposition,

$$\begin{aligned} & \left[ \frac{g_0(\epsilon)\nu_0^\epsilon}{4\pi} \right]^2 \Phi_{N,\epsilon}^{(l=1)}[\Gamma^{(0)pert}, \Gamma^{[r,0]}] \\ &= \Pi(\epsilon) \cdot \Delta_N^{[r]}(\{C(\{c\}^{[r]}, \{d\}^{[r]})\}^{[r]}, \{d\}_{N' \neq N}^{[r]}; \Lambda) \\ &+ \left[ \frac{g_0(\epsilon)\nu_0^\epsilon}{4\pi} \right]^2 \\ &\cdot \left\{ \Xi_N^{(1)}(\{c\}^{[r]}) \frac{1}{\epsilon} + \Gamma_N^{[r,1]}(\{c\}^{[r]}, \{d\}^{[r]}; \Lambda) + \mathcal{O}(\epsilon) \right\}. \end{aligned} \quad (1.6)$$

Here  $\{C\}^{[r]}$  is a set of nonlinear algebraic expressions in the input coefficients (1.3/1.4), while  $\{d\}_{N' \neq N}^{[r]}$  denotes the subset of denominator roots (1.4) in the vertices  $\Gamma_{N'}$  other than  $\Gamma_N$  to which  $\Phi_N$  provides coupling. The appearance of the first term on the r.h.s. of (1.6) is nontrivial: it occurs only if all seven basic vertices are treated by mutually consistent, nonperturbative approximants of the same level  $r$ . This term appears with a prefactor,

$$\Pi = \left[ \frac{g_0(\epsilon)}{4\pi} \right]^2 \frac{1}{\epsilon} \left( \frac{\Lambda_\epsilon^2}{\nu_0^2} \right)^{-\epsilon}, \quad (1.7)$$

which, by virtue of an exact RG identity, is independent of the renormalized coupling  $g(\nu)$ , and finite as  $\epsilon \rightarrow 0$ :

$$\Pi(\epsilon) = \frac{1}{\beta_0} [1 + \mathcal{O}(\epsilon, \epsilon \ln \epsilon)], \quad \text{independent of } g^2. \quad (1.8)$$

Here  $\beta_0$  is the leading beta-function coefficient of (1.1). Note how in this exact result one coupling factor ( $g_0^2$ ) and one divergence factor ( $\frac{1}{\epsilon}$ ) get ‘‘eaten’’ to produce a coupling-independent and finite quantity.

It therefore becomes possible to reproduce analytically the nonperturbative part,  $\Delta_N^{[r]}$ , of the zeroth-order input (1.5) by imposing the matching or self-consistency conditions,

$$\{d\}_{N' \neq N}^{[r]} = \{d\}_N^{[r]} \quad (\text{all } N), \quad (1.9)$$

which says that all basic vertices must exhibit one common set of denominator zeroes (still differing, however, for different types of external legs, or basic fields), and

$$\frac{1}{\beta_0} C_i^{[r]}(\{c\}^{[r]}, \{d\}^{[r]}) = c_{r,i} \quad (i = 1, \dots, k_r), \quad (1.10)$$

which ensures reproduction of numerator structures.

It is crucial that the nonperturbative terms establish themselves in a *finite* manner, since in this way one avoids the introduction of nonlocal counterterms, and thus preserves another basic element of standard renormalization. It is equally crucial that the above mechanism, as shown by the last two factors of (1.7), is *tied to the loop divergences* of the integral equations: this gives the superficially divergent vertices a privileged position, such that formation of nonperturbative  $\Delta_N$ 's remains rigorously restricted to these vertices. In spite of the infinite hierarchical coupling, *the number of extended Feynman rules does not proliferate*, and in fact remains the same as for the bare vertices.

In what follows, we focus exclusively on this self-consistency process for the generalized Feynman rules, and therefore refer the reader to [3] for what needs to be said about the last term of (1.6) – representing the  $p = 1$  quasi-perturbative correction – and about the perturbative “boundary condition” and essentially standard renormalization procedure it requires. We only note, for later use, that the condition of having the remaining divergence exactly equal to the perturbative one would require  $\Xi_N^{(1)}$  to be of the form

$$z_N^{(1)} I_N^{(0)pert}, \quad (1.11)$$

with  $z_N^{(1)}$  the one-loop coefficient of  $(g/4\pi)^2/\epsilon$  in the perturbative renormalization constant  $Z_N$  for the Vertex  $I_N$  – a condition which may impose extra constraints on  $\{c\}^{[r]}$  that for low  $r$  may be satisfiable only within approximation errors.

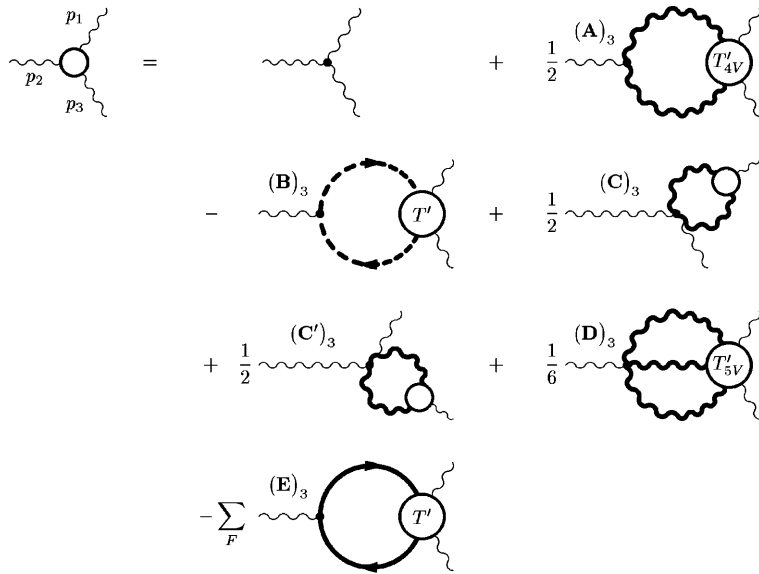
The exploration of this extended-iterative scheme represents a calculational program of some length. It was begun in [3] with an illustrative derivation of (1.6) and of matching conditions (1.10) at  $r = 1$  and  $l = 1$  for the transverse two-gluon vertex. While calculations of the ghost and fermion two-point functions follow essentially the same pattern, those for the higher superficially divergent vertices, with  $N = 3$  and 4, are not straightforward extensions to more kinematical variables. As a result of the subtle interplay between the coupled DS equations, they reveal a whole array of new aspects and intricacies. We therefore plan to present and discuss the  $r = 1, l = 1$  self-consistency calculations for these vertices in several parts. In the present paper, we focus on the remaining superficially divergent vertices of the gauge, ghost, and massless-quark sectors up to  $N = 3$ : the three-gluon vertex  $\Gamma_{3V}$ , ghost vertices  $\Gamma_{G\bar{G}}$  and  $\Gamma_{GV\bar{G}}$ , where  $V$  and  $G$  label vector (=gluon) and ghost external legs, and fermion vertices  $\Gamma_{F\bar{F}}$  and  $\Gamma_{FV\bar{F}}$ . In the companion paper [4] we will deal with the *four-gluon vertex*  $\Gamma_{4V}$ , the highest superficially divergent vertex, which is particularly complicated both kinematically and in its DS equation. In these two papers, consideration of the fermion (quark) functions will be restricted – as were the  $N = 2V$  calculations of [3] – to the case of massless quarks, where the fermionic mass scales,

too, are simply multiples of  $\Lambda$ . For massive fermions, the presence of “extraneous” RG-invariant mass scales not having the structure (1.1) causes additional complications with which we plan to deal separately.

In Sect. 2 of the present article, we recall the DS equation for the  $\Gamma_{3V}$  vertex. While summarizing known material, this section seems necessary to establish notation and a precise starting point for the subsequent discussion. In the present program we deal exclusively with the “ordinary” DS equations, without additional Bethe-Salpeter resummations in their interaction terms, in which the distinguished leftmost external line always runs into a *bare* vertex. Section 2.2 focuses on the phenomenon of compensating poles in the  $3V$  equation where they make their first appearance. While at first sight a merely technical point, these turn out to be an important structural element, by which the formalism automatically prevents the appearance of “wrong” poles on internal lines. A systematic account of these leads to the rearranged integral equation of Sect. 2.3, whose terms now exhibit an *extended-irreducibility property*. It is the rearranged equations that form the most convenient framework for the self-consistency problem of the extended Feynman rules. Extraction of the matching conditions (1.9-1.11) at the lowest level of rational approximation of interest for QCD ( $r = 1$ ) and at one loop ( $l = 1$ ) is discussed in Sect. 2.4 for the 3-gluon vertex. Section 3 considers the equations for the two ghost vertices, and Sect. 4 the equations for the two fermion vertices in the massless case.

The combined 2-plus-3-point self-consistency system is discussed in Sect. 5. A noteworthy result, which will continue to hold after inclusion of 4-gluon-conditions, is that the set of denominator coefficients (1.4), while restricted by (1.9), is not fully determined by the divergent parts of DS loops, and that composite-operator information, in the form of equation-of-motion condensate conditions (DS equations at coincident spacetime points) is required at this point to complement the usual equations. On the other hand, the system is found to nearly decouple from the 4-gluon one, so that a solution without 4-gluon equations is possible with an effective one-parameter freedom. The results, when compared to the more restricted and heuristic attempt of [5] for a pure-gluon theory, will be seen to represent significant progress, and in particular to include the existence of a parameter range where one of the several solutions to the nonlinear system has all zeroth-order vertices entirely real in the Euclidean, as required for a physical solution. One result that may be of interest is that the transverse gluon propagator of zeroth order (not including any  $O(g^2)$  corrections) at zero momentum is a finite constant of order  $\Lambda^{-2}$ .

As in [3], all calculations are performed *for the Euclidean theory, and in Landau gauge*. The Landau gauge provides some welcome reduction of the considerable complexity of loop computation with the extended Feynman rules: here the two ghost vertices turn out to remain perturbative, and calculations can be restricted to amplitudes with only transverse (if any) gluon legs, which then form a closed DS problem. This has the obvious disadvantage

Fig. 1. DS equation for the  $\Gamma_{3V}$  Vertex

that nothing can be inferred as yet about the approximate saturation or violation of Slavnov-Taylor identities, which we recall are statements about amplitudes with at least one *longitudinal* gluon leg, and which in the low orders of an iterative scheme are not, of course, expected to be exactly self-consistent. (Differential ST identities, under an additional regularity assumption, do constrain the transverse-gluon vertices to some extent, and this is discussed briefly at the end of Sect. 5). However, since the physical degrees of freedom of the gauge field are in the transverse sector, one may expect the essential parts of the nonperturbative structure to develop here (as is obviously true for the 2-gluon function). Also, we do not yet use the most general color-and-Lorentz-tensor structure of the 4V vertex, which would lead to calculations of prohibitive length, but restrict our study to a theoretically motivated tensor subset capable of dynamical self-consistency. Nor do we consider the quasi-perturbative corrections,  $(\frac{g}{4\pi})^2 \Gamma_N^{[1,1]}$  in the notation of (1.6). While all these questions are interesting in themselves, they must form subjects of future study.

To the extent that the present scheme uses the DS equations as a framework, its purpose is not to provide exact numerical solutions of DS equations at low levels of decoupling; for accounts of the work in this direction the reader is referred to [6]. Here the aim is to develop an analytic approximation method that provides some insight into the nonperturbative *coupling* structure, and in particular to identify the precise mechanism, connected with the divergence structure, by which the scale (1.1) establishes itself in correlation functions. In particular, such a scheme allows qualitative changes in the elementary propagators – the appearance of zeroth-order, finite, real or complex mass shifts – to be followed in a more transparent fashion.

## 2 The three-gluon vertex equation

### 2.1 Integral equation and input

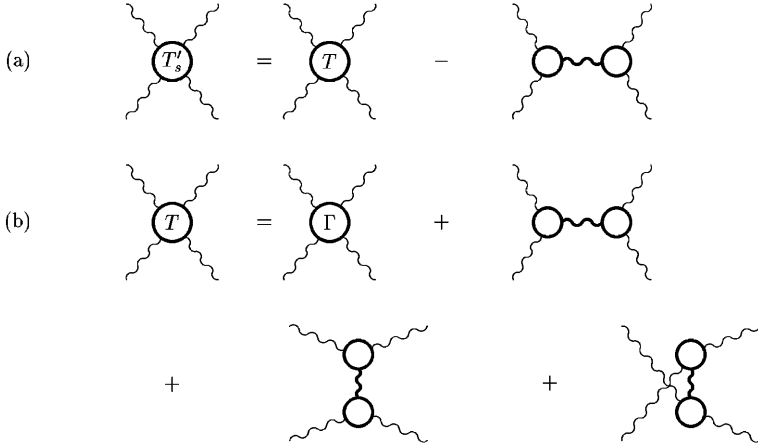
The DS equation for the proper three-vector vertex  $g_0 \Gamma_{3V}$  in Euclidean momentum space is written diagrammatically in Fig. 1. The form shown is a compact but hybrid one: most terms on the r.h.s. have not been resolved down to the level of proper vertices, but feature connected and amputated functions  $T'$  which are one-particle irreducible (1PI) only in the horizontal channel of the diagram, while otherwise still containing reducible (1PR) terms. Thus in term (A)<sub>3</sub> of Fig. 1, the four-gluon  $T$  matrix  $T'_{4V}$  for the horizontal channel is to be decomposed further as in Fig. 2:

$$T'_{4V} = T_{4V} - A_1 \quad (2.1)$$

$$T_{4V} = A_1 + A_2 + A_3 + \Gamma_{4V}. \quad (2.2)$$

Here  $\Gamma_{4V}$  is the proper, fully 1PI, four-gluon vertex, while  $A_1$  and  $A_2$ ,  $A_3$  are dressed one-gluon reducible terms in the horizontal channel and the two crossed channels, respectively. Analogous relations apply to the ghost-antighost-gluon-gluon and quark-antiquark-gluon-gluon  $T'$  matrices,  $T'_{G\bar{G}VV}$  and  $T'_{F\bar{F}VV}$ , of terms (B)<sub>3</sub> and (E)<sub>3</sub> respectively.

The “standard” form of the  $\Gamma_{3V}$  equation in Fig. 1 displays the characteristic asymmetry, common to all DS equations, of having the leftmost external leg always ending in a bare vertex, while the other legs run into dressed vertices. This structure is at the core of a problem plaguing all treatments (and not just the present approximation method) of vertices with  $N \geq 3$ : while the exact solution of the equation may be known to have a certain Bose or Fermi symmetry, the equation does not display this symmetry manifestly, and approximate solutions to it therefore usually fail to exhibit the full desired symmetry. Enforcing the symmetry by imposing extra conditions on the vertex coefficients leads to *overdetermination* in the


 Fig. 2. Decomposition of the four-gluon  $T$ -matrix

self-consistency equations. In the framework of an iterative solution, “trivial” symmetrizations may of course be used to cure this problem, but the unsymmetrized forms remain of interest as *indicators of the overall error* at the level of approximation considered.

Since we will be working throughout at the one-loop ( $l = 1$ ) level, characterized by a single  $D$ -dimensional momentum integration in the dressing functional  $\Phi_{3V}$ , the term  $(D)_3$  of Fig. 1 with two DS loops does not yet contribute (its contribution to the l.h.s. of (1.10) will be of order  $(1/\beta_0)^2$ ).

The *input* for the self-consistency calculation must consist of the Euclidean extended Feynman rules  $\Gamma_N^{[r,0]}$ , at the same level  $r$  of rational approximation, for all seven superficially divergent vertices. With the exception of  $\Gamma_{4V}^{[r,0]}$ , whose  $r = 1$  form will be detailed in [4], these have been listed in [3]. Here we need to recall only two elements carrying special restrictions. First, the gluon-propagator rule,  $D^{[r,0]} = -(\Gamma_{2V}^{[r,0]})^{-1}$ , will be simplified throughout by adopting the Landau ( $\xi = 0$ ) gauge fixing. Then

$$(D^{[r,0]}(k))^{\mu\nu} = \delta_{ab} t^{\mu\nu}(k) D_T^{[r,0]}(k^2), \quad (2.3)$$

$$t^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \quad (2.4)$$

where, at the  $r = 1$  level,

$$D_T^{[1,0]}(k^2) = \left[ k^2 + u_{1,1} A^2 + \frac{u_{1,3} A^4}{k^2 + u_{1,2} A^2} \right]^{-1} \quad (2.5)$$

$$= \frac{k^2 + u_{1,2} A^2}{(k^2 + \sigma_{1,1} A^2)(k^2 + \sigma_{1,3} A^2)}. \quad (2.6)$$

Second, the general *color* structure of the  $\Gamma_{3V}$  vertex itself, whose self-reproduction we examine in this section,

$$(\Gamma_{3V})_{abc} = i f_{abc} \Gamma_{(f)} + d_{abc} \Gamma_{(d)}, \quad (2.7)$$

will be simplified from the outset to a pure  $f_{abc}$  structure, i.e. one puts

$$\Gamma_{(d)} \approx 0 \quad (2.8)$$

and omits the  $f$  on  $\Gamma_{(f)}$ . The reason is that in the much more complicated color structure of  $\Gamma_{4V}$  discussed in [4], we will disregard those color-basis tensors that would feed the  $d_{abc}$  portion through the  $(A)_3$  term of Fig. 1. It is conceivable that some  $d_{abc}$  structure could be made to self-reproduce through the other one-loop terms of Fig. 1 alone, but since we view the seven basic DS equations as an interrelated whole, it does not seem consistent to us to keep one source of such terms and neglect the other. The tensor structure then is

$$\begin{aligned} (\Gamma_{3V}(p_1, p_2, p_3))_{abc}^{\rho\kappa\sigma} = & i f_{abc} \left\{ \delta^{\kappa\sigma} (p_2 - p_3)^\rho F_0(p_2^2, p_3^2; p_1^2) \right. \\ & + \delta^{\sigma\rho} (p_3 - p_1)^\kappa F_0(p_3^2, p_1^2; p_2^2) \\ & + \delta^{\rho\kappa} (p_1 - p_2)^\sigma F_0(p_1^2, p_2^2; p_3^2) \\ & + (p_2 - p_3)^\rho (p_3 - p_1)^\kappa (p_1 - p_2)^\sigma F_1(p_1^2, p_2^2, p_3^2) \\ & + [ 10 \text{ terms not contributing} \\ & \left. \text{to totally transverse vertex} \right\}. \end{aligned} \quad (2.9)$$

The terms in the last line do not enter the purely transverse DS system based on Landau gauge fixing that we use. Since almost all of the invariant-function relations following from the three-gluon Slavnov-Taylor identity involve at least one of the four additional invariant functions from these terms, they cannot be tested in the present calculation, which is natural since our system has no longitudinal gluons. The one exception is a relation following, under an additional regularity assumption on  $\partial\Gamma_{3V}/\partial p_i$  at  $p_i \rightarrow 0$ , from the differential ST identity, and which involves only the  $F_0$  function,

$$F_0(q^2, q^2; 0) = \frac{d}{dq^2} D_T^{-1}(q^2), \quad (2.10)$$

provided the ghost vertices remain perturbative. Our procedure generally will be not to impose such relations from the outset, as one does in calculations with fixed model vertices, but to let the self-consistency calculation run its course and check later for their fulfillment or violation.

At level  $r = 1$  (and only at  $r = 1$ ), the invariant functions  $F_0^{[1,0]}$  and  $F_1^{[1,0]}$  are conveniently written in a form

fully decomposed into partial fractions,

$$\begin{aligned} F_0^{[1,0]}(p_1^2, p_2^2; p_3^2) &= 1 + x_{1,1}(\Pi_1 + \Pi_2) \\ &+ \left(x_{1,2} + \frac{x'_{1,2}}{\Pi_3}\right)\Pi_1\Pi_2 + x_{1,3}\Pi_3 \\ &+ \left[\left(x_{1,4} + \frac{x'_{1,4}}{\Pi_2}\right)\Pi_1 + \left(x_{1,4} + \frac{x'_{1,4}}{\Pi_1}\right)\Pi_2\right]\Pi_3 \\ &+ x_{1,5}\Pi_1\Pi_2\Pi_3, \end{aligned} \quad (2.11)$$

$$\begin{aligned} F_1^{[1,0]}(p_1^2, p_2^2, p_3^2) &= \frac{1}{\Lambda^2} \left[ x_{1,6}(\Pi_1\Pi_2 + \Pi_2\Pi_3 + \Pi_3\Pi_1) \right. \\ &\left. + x_{1,7}\Pi_1\Pi_2\Pi_3 \right] \end{aligned} \quad (2.12)$$

featuring the building blocks

$$\Pi_i = \frac{\Lambda^2}{p_i^2 + u'_{i,2}\Lambda^2} \quad (i = 1, 2, 3). \quad (2.13)$$

Here we anticipate that the DS self-consistency conditions (1.9) will enforce  $u'_{1,2} = u_{1,2}$ , i.e., a common denominator factor in all gluonic vertices. The *factorized* (with respect to the 3 variables  $p_i^2$ ) denominator structure of these approximants may be viewed as arising from a triple-spectral representation,

$$F_{0,1} = \frac{1}{\pi^3} \int dz_1 dz_2 dz_3 \frac{\rho_{0,1}(z_1, z_2, z_3, p_1^2, p_2^2, p_3^2)}{(z_1 - p_1^2)(z_2 - p_2^2)(z_3 - p_3^2)}, \quad (2.14)$$

(where the spectral functions are still allowed some *polynomial*  $p_i^2$  dependence) by a discrete approximation. At  $r = 1$ , it exhibits a single denominator zero,  $p_i^2 = -u_{1,2}\Lambda^2$ , in all three variables.

Although the approximation of branch-cut structures by poles is an old technique, a casual look might still suggest a danger here that these poles in vertices could somehow take on a life of their own and roam the formalism as unphysical particles. This does not happen, for two reasons. First, DS self-consistency will transfer these poles down to the two-point vertices (negative-inverse propagators), so that the propagators themselves will develop *zeroes* at these pole positions, as seen in (2.6). The unamputated, connected Green functions, which are the quantities having physical interpretation as propagation amplitudes, will therefore be nonsingular at these positions in the squared momenta of their *external* lines, and no S-matrix elements for unphysical particles of masses  $u_{1,2}\Lambda^2$  will arise, as emphasized already in [3]. Second, and perhaps more remarkably, the formalism will also automatically cancel the poles of type (2.13) when they arise on certain *internal* lines, so that there will be no Cutkosky discontinuities corresponding to production of such objects. To demonstrate this in the simplest context will be the subject of the next subsection.

## 2.2 “Compensating” poles in $\Gamma_{4V}$

The phenomenon of automatic cancellation of superfluous poles on internal lines seems to have been noted first

by Jackiw and Johnson and by Cornwall and Norton [7] in 1973. While there are obvious technical differences between their (Abelian) models and the present QCD study (in particular, their poles arise in longitudinal-vector channels, whereas here they appear in the transverse-gluon sector), this mechanism is very much the same in both cases.

Consider the  $\Gamma_{3V}$  equation of Fig. 1 in order  $[r, 0]$ , and compare residues of both sides at the poles in the variable  $p_2^2$  of the leftmost leg. In a partial-fraction decomposition with respect to  $p_2^2$ , the l.h.s. can be written

$$\begin{aligned} \Gamma_{3V}^{[r,0]} &= \sum_{n=1}^r B_n^{[r]}(p_3, p_1) \frac{\Lambda^2}{p_2^2 + u_{r,2n}\Lambda^2} + B_0^{[r]}(p_3, p_1) \\ &+ [\text{terms with } (p_2^2)^1, (p_2^2)^2, \dots, (p_2^2)^r]. \end{aligned} \quad (2.15)$$

The compact notation suppresses all tensor structure. All terms displayed still have rational structure in  $p_3^2, p_1^2$ . The terms in the second line are allowed [3] as long as only the conditions of overall asymptotic freedom and preservation of perturbative power counting are imposed, but will in fact turn out to be more strongly restricted. The residue at  $p_2^2 = -u_{r,2n}\Lambda^2$  for the l.h.s. is then  $\Lambda^2 B_n$  ( $n = 1, \dots, r$ ).

For the r.h.s., to keep the argument as simple as possible, we invoke for the moment all simplifications available for our specific calculation according to subsect. 2.1, i.e. we disregard terms  $(D)_3, (E)_3$ , and  $(B)_3$  (the general case will be outlined in subsect. 2.3). Then *zeroth-order* denominator structure in  $p_2^2$  can come only from the  $T_{4V}^{[r,0]}$  amplitude of term  $(A)_3$ , for which  $p_2^2 = (p_3 + p_1)^2 = s_E$  is the (Euclidean) Mandelstam variable in the horizontal channel. According to general structure theorems on correlation functions [8], residues at poles in such a variable must factorize with respect to the two sides of the channel, or be at most sums of factorizing terms. Thus if  $q_1, q_3$  are the momenta of the loop gluons of term  $(A)_3$ , the  $T_{4V}'$  amplitude in the vicinity of  $s_E = -u_{r,2n}\Lambda^2$  must behave as

$$T_{4V}^{[r,0]} = \frac{\Psi_n^T(q_1, q_3)\Psi_n(p_3, p_1)}{s_E + u_{r,2n}\Lambda^2} + [\text{regular terms}]. \quad (2.16)$$

The partial Bose symmetry of  $T_{4V}'$  implies that both residue factors must be given by the same function or column vector of functions,  $\Psi_n$ . Note that (2.16) in no way contradicts the 1PI property of  $T_{4V}'$  in the horizontal channel: the pole factor is not a propagator of any of the *elementary* fields of the theory, so the term is technically allowed to appear (as would, e.g., a bound-state pole) not only in  $T_{4V}'$  but in fact in the fully 1PI piece,  $\Gamma_{4V}$ , of (2.2). However, the observation does suggest a natural enlargement of the notion of reducibility, as will be discussed below.

The residue comparison shows that  $\Psi_n$  must be proportional to  $B_n$ ,

$$\Psi_n(p_3, p_1) = M_n B_n(p_3, p_1), \quad (2.17)$$

with  $M_n$  some matrix, and that (again in compact notation)

$$\left\{ g_0^2 \frac{1}{2} \int \Gamma_{3V}^{(0)pert} DDB_n \right\}_{p_2^2 = -u_{r,2n}\Lambda^2} \cdot M_n^T M_n B_n = \Lambda^2 B_n. \quad (2.18)$$

But here the brackets on the l.h.s. are already fixed from a lower stage of the DS problem: the self-reproduction conditions for the poles of the gluon self-energy  $\Delta_{2V}(p_2)$ . Those enforce, under the simplifications adopted,

$$\left\{ g_0^2 \frac{1}{2} \int \Gamma_{3V}^{(0)pert} DDB_n \right\}_{p_2^2 = -u_{r,2n}\Lambda^2} = -u_{r,2n+1} \Lambda^2 t(p_2), \quad (2.19)$$

$t$  being the transverse projector of (2.4), and  $u_{r,2n+1}$  the dimensionless residue parameter in  $\Gamma_{2V}^{[r,0]}$  exemplified, for  $r = 1$ , by the  $u_{1,3}$  of (2.5). We conclude that, first,

$$B_n^{[r]}(p_3, p_1) = t(p_3 + p_1) B_n^{[r]}(p_3, p_1) \quad (n = 1, \dots, r), \quad (2.20)$$

an information more detailed than that of the all-transverse projection (2.9): nonperturbative denominator structure develops only in the variables of *transverse* gluon legs. Second,

$$M_n^T M_n = -\frac{1}{u_{r,2n+1}}. \quad (2.21)$$

What has been learned is that *as a consequence of the lower* (two-point and three-point) *equations alone*,  $T_{4V}^{[r,0]}$  and  $\Gamma_{4V}^{[r,0]}$  must contain a term of the form

$$-(C_1^{[r]})_{2V,2V} = -\sum_{n=1}^r B_n^{[r]} \frac{(u_{r,2n+1})^{-1} t(P)}{P^2 + u_{r,2n}\Lambda^2} B_n^{[r]} \quad (2.22)$$

in the total momentum  $P$ , with  $P^2 = s_E$ , of its horizontal channel. But  $\Gamma_{4V}$  is fully Bose symmetric, and therefore must contain analogous terms also for the two crossed channels, with Mandelstam variables  $u_E$  and  $t_E$ , which we denote by  $C_2$  and  $C_3$ . Thus,

$$\begin{aligned} \Gamma_{4V}^{[r,0]} = & -(C_1^{[r]})_{2V,2V} - (C_2^{[r]})_{2V,2V} \\ & - (C_3^{[r]})_{2V,2V} + V_{4V}^{[r,0]}, \end{aligned} \quad (2.23)$$

and the derivation shows that the  $V_{4V}^{[r,0]}$  defined by this relation contains no more *zeroth-order* denominator structure in  $s_E$ ,  $u_E$ , or  $t_E$ . The latter conclusion, strictly speaking, follows only for the adjoint color representation in each two-body channel, since in Fig. 1 the leftmost gluon line projects  $T'_{4V}$  onto this color subspace. It requires additional considerations, based on the Bethe-Salpeter normalization conditions, to check for possible zeroth-order, Mandelstam-variable poles in the other colored channels. Since these considerations logically belong to the discussion of the four-gluon amplitude, we defer them to the companion article [4]. Here we anticipate the result: *In zeroth perturbative order*, there are no Mandelstam-variable poles in the other color sectors. Thus the “reduced” vertex function  $V_{4V}^{[r,0]}$  has nonperturbative structure, rationally approximated, only in the variables  $p_1^2, \dots, p_4^2$  of individual external legs. We emphasize that this in no way precludes the existence of glueball-type bound states in color-singlet

channels. Such bound states are not elements of the generalized Feynman rule  $\Gamma_{4V}^{[r,0]}$  at any level  $r$ , but arise through the standard mechanism of partial (e.g., ladder) resummation of quasi-perturbative corrections  $(g/4\pi)^{2p} \Gamma_{4V}^{[r,p]}$  to all orders  $p \geq 1$ .

The poles of (2.16) cannot represent bound states, since in general their residues are not positive definite (at  $r = 1$ , for example,  $u_{1,3}$  will be found to be positive). Thus for the moment they would seem to be unphysical artefacts. But one immediately realizes that in fact they play a legitimate role by cancelling another unphysical phenomenon. Inspect the analytic structure of the one-gluon-reducible terms  $A_i$  at level  $[r, 0]$  (i.e., with  $[r, 0]$  diagram elements). Again suppressing all unnecessary arguments, one has

$$A_1^{[r,0]} = \Gamma_{3V}^{[r,0]} D^{[r,0]}(P) \Gamma_{3V}^{[r,0]}. \quad (2.24)$$

By the construction prescriptions for the extended Feynman rules, the gluon propagator  $D^{[r,0]}$  contains, besides its  $r + 1$  poles, a product of  $r$  numerator zeroes of the form

$$\prod_{n=1}^r (P^2 + u_{r,2n}\Lambda^2). \quad (2.25)$$

On the other hand, both  $\Gamma_{3V}^{[r,0]}$  vertices contain the same product in their denominators, so that there remain on the internal gluon line, in addition to the legitimate poles from  $D$  describing propagation of the exchanged object, a number  $r$  of extra poles at positions  $P^2 = -u_{r,2n}\Lambda^2$ . Such extra poles are unacceptable physically; they would imply that the generalized Feynman rule for  $D$  is incomplete. Now isolate the  $n$ -th unphysical-pole piece of  $A_1$ . From (2.15) and (2.19), it must involve  $B_n t(P) B_n$ ; by computing the residue, the piece is found to be

$$B_n^{[r]} \frac{(u_{r,2n+1})^{-1} t(P)}{P^2 + u_{r,2n}\Lambda^2} B_n^{[r]}, \quad (2.26)$$

so that the sum of the unphysical pieces is the *negative* of (2.22):

$$\sum_{n=1}^r (n\text{-th unphysical pole of } A_1^{[r,0]}) = (C_1^{[r]})_{2V,2V}. \quad (2.27)$$

Upon combining (2.2) and (2.23) into

$$T_{4V}^{[r,0]} = A_1^{[r,0]} + A_2^{[r,0]} + A_3^{[r,0]} + V_{4V}^{[r,0]}, \quad (2.28)$$

the unphysical artefacts then cancel exactly to leave the “softened” exchange graphs

$$A_i^{[r,0]} = A_i^{[r,0]} - (C_i^{[r]})_{2V,2V} \quad (i = 1, 2, 3). \quad (2.29)$$

The  $\Gamma_{4V}^{[r,0]}$  poles inferred through the  $2V$  and  $3V$  equations therefore turn out to be “compensating poles”, cancelling unphysical parts in the one-gluon-reducible terms

(a)  $\text{Diagram with double wiggly line and ghost loop} = \text{Diagram with single wiggly line and ghost loop} - \text{Diagram with ghost loop and triangle}$

(b)  $\text{Diagram with ghost loop and triangle} = \sum_{n=1}^r B_n^{[r]} \frac{(u_{r,2n+1})^{-1} t(P)}{P^2 + u_{r,2n} \Lambda^2} B_n^{[r]}$

**Fig. 3.** “Softened” exchange diagram (a) and diagrammatic representation of the compensating pole (b)

of  $T_{4V}$ . It is clear that in the context of the extended iterative scheme it is the artefact-free  $A'_i$ , rather than the original  $A_i$ , that represent the *physical* one-gluon exchange mechanism, and the artefact-free  $V_{4V}^{[r,0]}$ , rather than  $\Gamma_{4V}^{[r,0]}$ , that constitutes the physical extended Feynman rule for four-gluon interaction. On the other hand, in the quantity (2.1),

$$T'_{4V}{}^{[r,0]} = A'_2{}^{[r,0]} + A'_3{}^{[r,0]} + V_{4V}{}^{[r,0]} - (C_1^{[r]})_{2V,2V}, \quad (2.30)$$

an uncompensated  $C_1$  remains, enabling, as we have seen, the  $(A)_3$  term of Fig. 1 to produce the  $B_n$  terms of (2.15).

### 2.3 Extended irreducibility. Rearranged vertex equation

In the general case where terms  $(B)_3$ ,  $(E)_3$ , and (for  $l \geq 2$  loops)  $(D)_3$  contribute, one first generalizes (2.16) to the conclusion that the  $T$  matrices  $T'_{G\bar{G}VV}$ ,  $T'_{F\bar{F}VV}$ ,  $T'_{5V}$  in general have poles (forming discrete approximations of cuts) at the same positions,  $p_2^2 = -u_{r,2n}\Lambda^2$ , in their horizontal channels. One then invokes another general property of correlation functions [8]: when a pole is present in several functions simultaneously, each residue factor is *uniquely* associated with its own subset of external lines, and independent of the remaining legs in the various functions. Thus at the pole  $p_2^2 = -u_{r,2n}\Lambda^2$ , the residue factor for the rightmost two-gluon configuration in Fig. 1 is the same  $\Psi_n$  as in (2.16) for all  $T'$  amplitudes. Then (2.17) still follows, but (2.18) now has four different contributions on its l.h.s., corresponding to various dressing mechanisms for the gluon self-energy. Self-reproduction of the  $B_n$  terms with  $n \geq 1$  in (2.15) is possible if  $M_n$  and the various matrices taking the place of the  $M_n^T$  in (2.18) are all the same multiple of  $t(p_2)$ . Then the full two-gluon self-consistency conditions [3] can be invoked, which generalize (2.19), and the  $B_n$  term reproduces itself if (2.21) is imposed.

As a by-product, one finds that the 1PI functions  $\Gamma_{G\bar{G}VV}$ ,  $\Gamma_{F\bar{F}VV}$  must contain terms of the form

$$-(C_1^{[r]})_{G\bar{G},VV} = -\sum_{n=1}^r \tilde{B}_n^{[r]} \frac{(u_{r,2n+1})^{-1} t(P)}{P^2 + u_{r,2n} \Lambda^2} B_n^{[r]}, \quad (2.31)$$

$$-(C_1^{[r]})_{F\bar{F},VV} = -\sum_{n=1}^r \bar{B}_{F,n}^{[r]} \frac{(u_{r,2n+1})^{-1} t(P)}{P^2 + u_{r,2n} \Lambda^2} B_n^{[r]}, \quad (2.32)$$

in their two-body channels  $(G + \bar{G} \leftrightarrow V + V)$  and  $(F + \bar{F} \leftrightarrow V + V)$  respectively, where  $\tilde{B}_n$ ,  $\bar{B}_{F,n}$  are the amplitudes analogous to the  $B_n$  of (2.15) in the partial-fraction decompositions of the ghost vertex  $\Gamma_{GV\bar{G}}^{[r,0]}(-q', k, q)$  and

fermion vertices  $\Gamma_{FV\bar{F}}^{[r,0]}(-p', k, p)$  with respect to their gluon-leg variable,  $k^2$ . The analogous but richer structure in  $T'_{5V}$  will be discussed in detail in [4].

The structure revealed by these residue-taking operations may look involved at first, but the final result is simple: the full 4-gluon, off-shell  $T$  matrix (2.2), for example, has no unphysical artefacts at all. The artefacts arose because, in a nonperturbative context, the usual decomposition of  $T$  by the criterion of ordinary one-particle (here, one-gluon) reducibility turns out to be an awkward one: both parts in such a division contain unphysical-pole terms that cancel in the sum. It is clearly more natural, and better suited to the physics of the problem, to perform the decomposition as in (2.28), where all parts are free of artefacts. To characterize such a decomposition more formally, we call the set of  $r$  pole factors common to expressions (2.22) and (2.31/2.32) a gluonic shadow, described graphically by the double wiggly line of Fig. 3, and define as one-shadow-irreducible any amplitude built from  $[r, 0]$  extended Feynman rules that does not fall into two disconnected pieces upon cutting such a shadow line. The defining property of decomposition (2.28) then is that *all* its terms are one-shadow irreducible. In particular,  $V_{4V}^{[r,0]}$  exhibits what one may call *extended irreducibility*, being irreducible both for gluon-propagator poles and for shadow poles, while the “softened” exchange diagrams  $A'_i{}^{[r,0]}$ , described graphically by using dotted diagram elements as in Fig. 3, are still reducible for the gluon-propagator poles.

Use of (2.30) and of the analogous decompositions

$$T'_{G\bar{G}VV}{}^{[r,0]} = \tilde{A}'_2{}^{[r,0]} + \tilde{A}'_3{}^{[r,0]} + V_{G\bar{G}VV}{}^{[r,0]} - (C_1^{[r]})_{G\bar{G},VV}, \quad (2.33)$$

$$T'_{F\bar{F}VV}{}^{[r,0]} = \bar{A}'_{F,2}{}^{[r,0]} + \bar{A}'_{F,3}{}^{[r,0]} + V_{F\bar{F}VV}{}^{[r,0]} - (C_1^{[r]})_{F\bar{F},VV}, \quad (2.34)$$

where  $\tilde{A}'_i$  and  $\bar{A}'_{F,i}$  have obvious meanings as softened exchange diagrams reducible for ghost or fermion propagator poles but with ghost-shadow (in non-Landau gauges only) or quark-shadow poles compensated, now leads to the rearranged vertex equation of Fig. 4. With the pole terms of (2.15) having been reproduced on the r.h.s. through condition (2.21) on the *four*-gluon function, and with the second line of (2.15) anticipated to be absent by restriction (2.38) below, only an equation for the amplitude  $B_0^{[r]}$  – an object with three-gluon tensor structure but only two scalar variables – remains. This rather strong reduction of the original equation, due to the “all-in-one-blow” self-reproduction of the  $B_n$  parts with  $n \geq 1$  through the presence of the compensating poles, generally causes a loss of self-consistency conditions and therefore *underdetermi-*



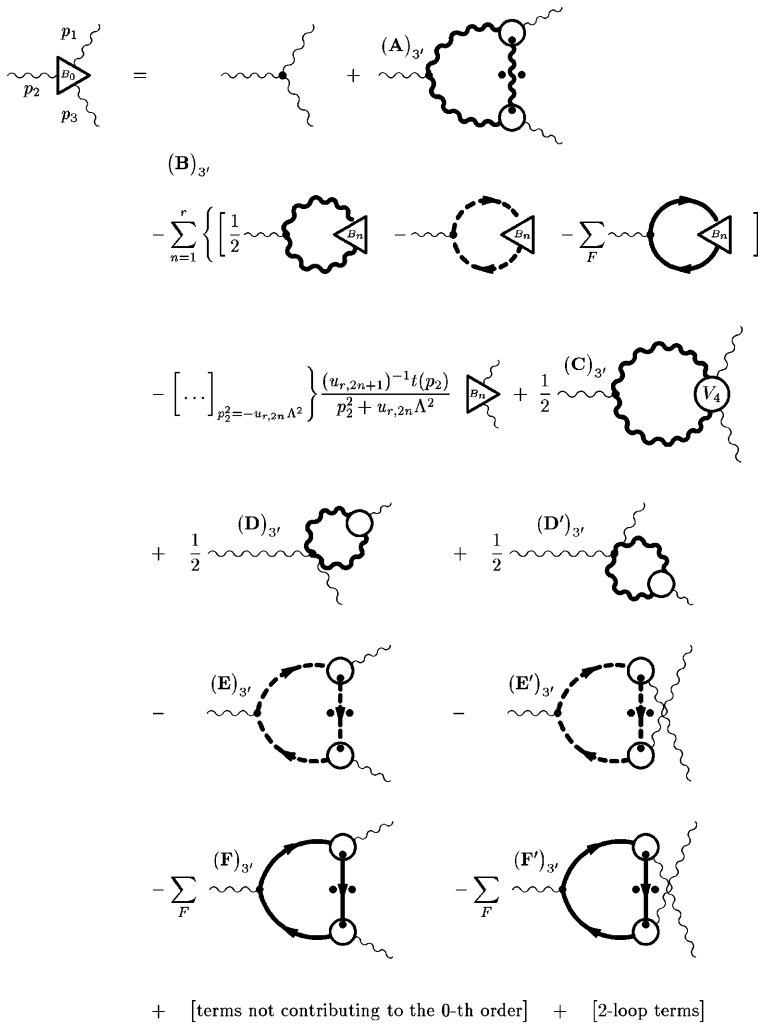


Fig. 4. DS equation for the  $B_0$  part of the  $\Gamma_{3V}$ -Vertex

nation, which tends to counteract the overdetermination coming from the lack of manifest symmetry. (At  $r = 1$ , for example, the  $x_{1,5}$  coefficient of (2.11) appears only in  $B_1^{[1]}$  and not in  $B_0^{[1]}$ , and therefore gets no self-consistency condition of its own).

In Fig. 4, we now obtain the dot modifications, signaling absence of unphysical artefacts, in the triangle diagrams  $(A)_{3'}$ ,  $(E, E')_{3'}$ , and  $(F, F')_{3'}$ . In the remainder, only the terms that can contribute to the self-reproduction of the extended Feynman rule (i.e., produce terms of zeroth perturbative order) at one loop have been made explicit. Thus diagrams containing  $V_{G\bar{G}VV}$  or  $V_{F\bar{F}VV}$  no more appear: for these amplitudes, which are superficially convergent, it is now true that after extraction of the shadow-pole terms they consist only of superficially convergent loops. Therefore they cannot develop zeroth-order terms through the divergence-related self-consistency mechanism, but remain at least of order  $g^2$ :

$$V_{G\bar{G}VV} = O(g^2); \quad V_{F\bar{F}VV} = O(g^2). \quad (2.35)$$

For their contributions to the 3-gluon DS equation, the argument of [3] goes through : their insertion into terms  $(B)_3$  and  $(E)_3$  of Fig. 1 produces integrals for which the

number of  $\frac{1}{\epsilon}(A_\epsilon^2/\nu_0^2)^{-\epsilon}$  factors lags behind the number of  $g_0^2$  prefactors by at least one, and which therefore give only quasi-perturbative corrections of order  $p \geq 1$ . Note that this would not have been true for the corresponding  $\Gamma$  amplitudes before extraction of the nonperturbative shadow pieces.

We see here that the argument of [3] concerning these higher amplitudes needs a subtle qualification: that argument did not take into account the possibility of certain *treelike* structures, of zeroth perturbative order, which nevertheless appear in the 1PI functions. The shadow-pole terms are precisely such structures. As is clear already from (2.33) and (2.34), they will appear in all higher, superficially convergent vertex functions, and are expected to follow a pattern of one shadow term arising for each one-particle reducible term of the corresponding  $T$ -matrix amplitude to provide compensation of its unphysical poles. We will prove this in detail in [4] for the representative case of the lowest superficially convergent amplitude in the gluon sector, the five-gluon amplitude, which already features a large and varied assortment of 1PR terms. Here we reemphasize that presence of these shadow terms does *not* imply a proliferation of Feynman rules, since they con-

sist entirely of building blocks determined already at the level of the basic vertices. As in  $V_{4V}$ , their role in the end is to establish that in the nonperturbative context, partitioning of  $T$  matrices should be performed according to an *extended* irreducibility criterion if the parts are to represent physical mechanisms not contaminated by unphysical-pole terms. For the  $V$  amplitude defined by such a decomposition, it is then generally true that

$$V_N^{[r,0]} = 0 \text{ (all } r), \text{ if } \Gamma_N \text{ superficially convergent,} \quad (2.36)$$

with eqs. (2.35) as special cases. The superficially divergent  $V_{4V}$  occupies a special position in that it is the only  $V$  amplitude with nonvanishing zeroth-order terms, approximated by the sequence  $V_{4V}^{[r,0]}$ . (The three-point vertices, having no 1PR terms, need no distinction between  $\Gamma_3$  and  $V_3$ ).

For the self-consistency problem, the rearrangement for shadow irreducibility brings both simplifications and complications. On the one hand we now obtain tighter restrictions than before on the structure of vertex approximants. Numerator polynomials of approximants to, e.g., the invariant function  $F_0$  of the three-gluon vertex (2.9) are of the general form

$$\sum_{\lambda, m_1, m_2, m_3 \geq 0} C_{\lambda m_1 m_2 m_3}^{[r]} (p_1^2)^{m_1} (p_2^2)^{m_2} (p_3^2)^{m_3} (\Lambda^2)^\lambda, \quad (2.37)$$

and asymptotic freedom, which governs the situation where all three momenta are scaled uniformly, gives only the restriction  $m_1 + m_2 + m_3 = 3r - \lambda \leq 3r$ . Consideration of situations where the vertex (2.9) sits at the outer edge of an 1PI diagram, so that only two of its momenta are running in loops while the third is external and fixed, sharpens this to  $m_i + m_j \leq 2r$  for all pairs  $i \neq j$  if perturbative degrees of loop divergence are to be preserved [3]. Now, more restrictions arise from the fact that in the decomposition (2.23) we are forced to require the  $C_i^{[r]}$  terms and the  $V_{4V}^{[r,0]}$  term *separately* to maintain perturbative degrees of divergence. Briefly, this conclusion follows from the fact that the  $\Gamma_{4V}^{[r,0]}$  on the l.h.s. does preserve the perturbative degrees by construction, and does so in two different “outer-edge” situations, with (i) one of the four legs being external and (ii) two legs being external. In both situations the  $B_n^{[r]}$  factors present in the  $C_i^{[r]}$  by (2.22), when restricted only by  $m_i + m_j \leq 2r$ , cause higher-than-perturbative degrees of divergence which in fact have the pathological property of increasing with  $r$ . One verifies by inspection that due to the different structure of both contributions, admitting terms in  $V_{4V}^{[r,0]}$  to compensate for those divergences in the first situation fails to cancel them in the second situation, and vice versa. The only way of making (2.23) consistent is therefore to exclude the “dangerous” momentum powers in each  $C_i$ . This forbids terms with net positive powers of  $p_3^2$  or  $p_1^2$  in the  $B_n^{[r]}$  factors, or equivalently, removes the terms in the second line of the partial-fraction decomposition (2.15). In the numerator

polynomial for  $F_0$  we therefore have the stronger restrictions

$$m_1 \leq r, \quad m_2 \leq r, \quad m_3 \leq r. \quad (2.38)$$

Such terms are also the only natural ones from the point of view of the spectral representation (2.14); they correspond to restricting the extra polynomial  $p_i^2$  dependence of the spectral functions in (2.14) to at most a bilinear one. At  $r = 1$ , in particular, this restriction forces the primed coefficients of (2.11) to zero:

$$x'_{1,2} = 0 \quad ; \quad x'_{1,4} = 0. \quad (2.39)$$

With  $\Gamma_{3V}$ 's thus simplified, the gluon-exchange graphs  $A_i$  like (2.24), and therefore the “softened” exchange graphs (2.29), now contain only terms decreasing like  $Q^{-2}$  in the momentum  $Q$  of their exchanged gluon. Therefore they can now be obtained simply by enumerating and adding the physical gluon-propagator poles with their residues: in Landau gauge fixing,

$$\begin{aligned} A_1^{[r,0]}(k_1, \dots, k_4) = & \sum_{m=0}^r [\Gamma_{3V}(k_1, k_2, P)]_{P^2 = -\sigma_{r,2m+1}\Lambda^2} \frac{\rho_m t(P)}{P^2 + \sigma_{r,2m+1}\Lambda^2} \\ & \times [\Gamma_{3V}(P, k_3, k_4)]_{P^2 = -\sigma_{r,2m+1}\Lambda^2}, \end{aligned} \quad (2.40)$$

where

$$\begin{aligned} \rho_m = & [(P^2 + \sigma_{r,2m+1}\Lambda^2)D_T(P)]_{P^2 = -\sigma_{r,2m+1}\Lambda^2} \\ = & \frac{\prod_{n=1}^r (u_{r,2n} - \sigma_{r,2m+1})}{\prod_{\substack{n=0 \\ (n \neq m)}}^r (\sigma_{r,2n+1} - \sigma_{r,2m+1})}, \end{aligned} \quad (2.41)$$

and where  $P = k_1 + k_2 = -(k_3 + k_4)$ . In non-Landau gauges, of course, the longitudinal-gluon propagator term of the original  $A_1$  must be added unchanged.

On the other hand the rearrangement leads to the result (which is true generally but was not yet visible in the simpler case of the two-point equation at one loop treated in [3]) that the perturbative limit as  $\Lambda \rightarrow 0$  cannot, at low levels  $r$ , be maintained exactly in all amplitudes, but only asymptotically for increasing  $r$ . For  $\Lambda \rightarrow 0$ , the  $V_{4V}^{[r,0]}$  of (2.23), as well as the  $A_i^{[r,0]}$  of (2.29), go over into their zeroth-order perturbative counterparts, but the  $C_i^{[r]}$  for low  $r$  do not go to zero:

$$\begin{aligned} & [(C_1^{[r]})_{2V,2V}]_{\Lambda=0} \\ & = \frac{1}{P^2} \left\{ \sum_{n=1}^r B_n^{[r]}(\Lambda=0) \frac{t(P)}{u_{r,2n+1}} B_n^{[r]}(\Lambda=0) \right\}. \end{aligned} \quad (2.42)$$

Thus diagrams involving the one-gluon-exchange mechanisms  $A'_i, \bar{A}'_i, \bar{A}'_{F,i}$ , such as the triangle diagrams of Fig. 4, are expected not to exhibit a fully correct perturbative limit as  $\Lambda \rightarrow 0$  at low levels  $r$ . The condition (or rather

conditions, because there are several tensor structures involved) for the curly bracket in (2.42) to vanish, of which we will present examples, can at best be fulfilled asymptotically for large  $r$ , where they spread their restrictive effect over an increasing number of nonperturbative vertex coefficients, and thus become progressively easier to maintain.

## 2.4 3-gluon, $r = 1$ self-consistency conditions

To extract the self-reproduction conditions of the extended Feynman rule  $\Gamma_{3T}^{[1,0]}$  one evaluates, with eqs. (1.7/1.8) in mind, the divergent parts of the terms on the r.h.s. of Fig. 4 in dimensional regularization and Landau gauge fixing, with  $[1, 0]$  diagram elements throughout. The restriction to divergent parts makes these calculations somewhat analogous to the computation of one-loop renormalization constants in perturbation theory (and much more feasible than full evaluations of  $[1, 1]$  radiative corrections, which already at  $r = 1$  are very lengthy). For the input  $V_{4T}^{[1,0]}$  to diagram  $(C)_{3'}$ , we anticipate formulas from appendix B of the companion article [4]: this approximant represents a theoretically motivated restriction to a subset of fifteen of the many possible color and Lorentz tensor structures of a four-gluon amplitude, with invariant functions characterized by a set  $\zeta = \{\zeta_1, \dots, \zeta_{17}\}$  of seventeen dimensionless, real numerator coefficients. The terms proportional to the number  $N_F$  of quark flavors, arising from fermion-loop diagrams  $(F)_{3'}$  and  $(F')_{3'}$ , are valid for *massless quarks* ( $\hat{m}_F = 0$  in the notation of the appendix of [3]), where all fermionic mass scales, too, are simply multiples of  $\Lambda$ . For brevity, we abstain from listing contributions of the various diagrams separately [9] and present only the combined results. Eqs. (1.10) for the coefficients  $x_{1,i}$  (now written  $x_i$  for brevity) of the  $F_0^{[1,0]}$  invariant functions (2.11) are:

$$\frac{1}{\beta_0} \left[ -\frac{9}{4}x_1 + \frac{15}{16}x_3 + \frac{1}{u_3} \left( \frac{1}{4}x_1x_2 - 9x_1x_4 + x_3x_4 \right) + \frac{2}{3}N_F \left( z_3 + \frac{1}{u_3}x_4z_3 - \frac{1}{w_3}z_1z_4 \right) \right] = x_1 \quad (2.43)$$

$$\frac{1}{\beta_0} \left[ \frac{3}{2}x_3^2 + \frac{1}{u_3} \left( \frac{1}{2}x_2x_4 - 2x_4^2 - \frac{15}{2}x_1x_5 + \frac{5}{4}x_3x_5 \right) + \frac{2}{3}N_F \left( z_3^2 + \frac{1}{u_3}x_5z_3 - \frac{1}{w_3}z_4^2 \right) \right] = x_2 \quad (2.44)$$

$$\frac{1}{\beta_0} \left[ \frac{3}{2}x_3 + \frac{1}{u_3} \left( -\frac{37}{4}x_1x_4 + \frac{3}{2}x_3x_4 \right) - Z_1(\zeta) + \frac{2}{3}N_F \left( z_3 + \frac{1}{u_3}x_4z_3 - \frac{1}{w_3}z_1z_4 \right) \right] = x_3 \quad (2.45)$$

$$\frac{1}{\beta_0} \left[ -\frac{9}{4}x_1 + \frac{15}{16}x_3 + \frac{1}{u_3} \left( -\frac{31}{4}x_1x_2 - \frac{5}{4}x_1x_4 + \frac{5}{4}x_2x_3 \right) - Z_1(\zeta) + \frac{2}{3}N_F \left( z_3 + \frac{1}{u_3}x_2z_3 - \frac{1}{w_3}z_1z_4 \right) \right] = x_1 \quad (2.46)$$

$$\frac{1}{\beta_0} \left[ \frac{3}{2}x_3^2 + \frac{1}{u_3} \left( -\frac{1}{4}x_2x_4 - \frac{5}{4}x_4^2 - \frac{15}{2}x_1x_5 + \frac{5}{4}x_3x_5 \right) - Z_2(\zeta) + \frac{2}{3}N_F \left( z_3^2 + \frac{1}{u_3}x_5z_3 - \frac{1}{w_3}z_4^2 \right) \right] = x_4 \quad (2.47)$$

(The fermionic vertex coefficients  $z_i$  are defined by (4.6) below.)

On the other hand, the terms of the invariant function  $F_1$  of (2.12) turn out to be fed only by themselves, and by terms in  $V_{4V}$  which our above-mentioned, restricted form of that amplitude omits in the first place. Again, since we view the basic vertices as an interrelated whole, it did not seem consistent to us to keep just one source of such terms. Thus

$$x_6 = 0 \quad ; \quad x_7 = 0 \quad (2.48)$$

is a consistent and *self-consistent* choice in our framework.

A noteworthy feature is that the coupling to the 4-gluon amplitude enters only into the three equations (2.45-2.47), and only through two linear combinations of its seventeen coefficients  $\zeta$ ,

$$Z_1(\zeta) = \frac{15}{32} (3\zeta_1 - \zeta_7), \quad (2.49)$$

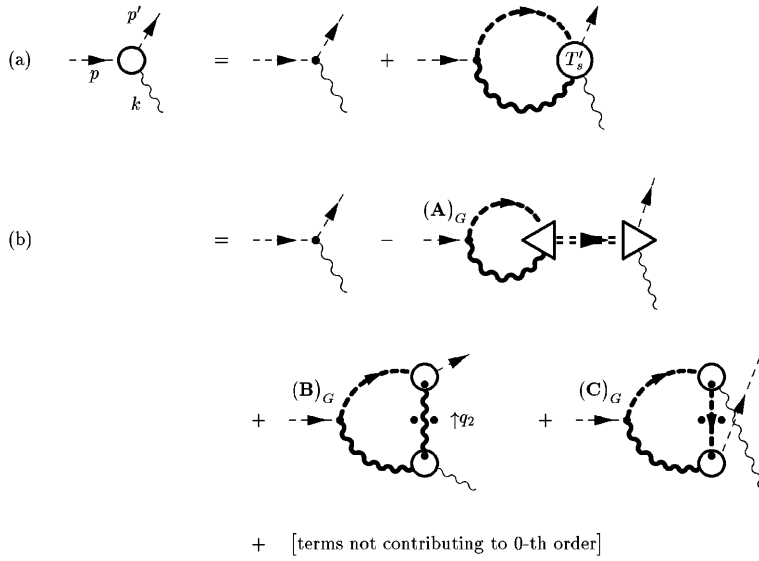
$$Z_2(\zeta) = \frac{15}{32} (3\zeta_2 + 3\zeta_3 - \zeta_8 - \zeta_9). \quad (2.50)$$

The observation that the rather large (and, as it will turn out, strongly overdetermined) self-consistency problem of the four-point vertex parameters couples to the 2-point and 3-point problems only through this narrow ‘‘bottle-neck’’ will be important as it will suggest ways of breaking down the rather voluminous total self-consistency problem into more manageable pieces.

As already noted, there is no equation with  $x_5$  on its r.h.s. But  $B_0^{[1]}$  still has three-gluon tensor structure, and  $x_1$ , by (2.9), appears twice in conjunction with two different tensor structures, so there are *two* equations, (2.43) and (2.46), for  $x_1$ . The relation obtained by subtracting these,

$$\frac{1}{u_3} \left( 8x_1x_2 - \frac{31}{4}x_1x_4 + x_3x_4 - \frac{5}{4}x_2x_3 \right) + Z_1(\zeta) - \frac{2}{3}N_F \left( \frac{x_2 - x_4}{u_3} \right) z_3 = 0, \quad (2.51)$$

represents the imposition, in zeroth perturbative order, of Bose symmetry on a DS equation that is not manifestly Bose symmetric. It appears to be fortuitous that the ‘‘loss’’ of one equation, incurred in the reduction of the self-consistency problem to the partial amplitude  $B_0^{[1]}$ , is just compensated by one Bose-symmetry restriction; we are not aware of a deeper reason for this phenomenon.



**Fig. 5.** DS equation for the ghost-gluon 3-point vertex in the ghost channel

Finally we note the result for the coefficient of the perturbative-remainder divergence, the  $\Xi_N^{(1)}$  of (1.6):

$$\begin{aligned}
\left(\Xi_{3T}^{(1)}\right)^{\rho\kappa\sigma} = & \\
& \delta^{\kappa\sigma}(p_2 - p_3)^\rho \left[ -\frac{17}{4} + \frac{1}{u_3} \left( -\frac{37}{4}x_1^2 + \frac{3}{2}x_1x_3 \right) \right. \\
& + \frac{2}{3}N_F \left( 1 + \frac{1}{u_3}x_1z_3 - \frac{1}{w_3}z_1^2 \right) \\
& + \delta^{\sigma\rho}(p_3 - p_1)^\kappa \left[ -\frac{17}{4} + \frac{1}{u_3} \left( -x_1^2 - 8x_1x_3 + \frac{5}{4}x_3^2 \right) \right. \\
& + \frac{2}{3}N_F \left( 1 + \frac{1}{u_3}x_3z_3 - \frac{1}{w_3}z_1^2 \right) \\
& + \delta^{\rho\kappa}(p_1 - p_2)^\sigma \left[ -\frac{17}{4} + \frac{1}{u_3} \left( -\frac{37}{4}x_1^2 + \frac{3}{2}x_1x_3 \right) \right. \\
& + \frac{2}{3}N_F \left( 1 + \frac{1}{u_3}x_1z_3 - \frac{1}{w_3}z_1^2 \right) \left. \right] \quad (2.52)
\end{aligned}$$

The purely perturbative result would be of the form (1.11) with the Landau-gauge value

$$z_{3V}^{(1)}(\xi = 0) = -\frac{17}{4} + \frac{2}{3}N_F. \quad (2.53)$$

The existence of deviations from the perturbative limit does not come as a surprise in view of what we noted before in connection with (2.42). Eliminating them would again produce overdetermination of the  $x$  coefficients and generally is not feasible exactly for low  $r$  but only asymptotically for large  $r$ . However the fact that they also cause deviations from the full Bose symmetry of the 1-loop divergence (more precisely, they leave only the partial symmetry exemplified by diagram  $(A)_{3'}$  of Fig. 4) is again tied to the lack of manifest Bose symmetry in  $N \geq 3$  DS equations, and is not specific to the present method but will plague all truly nonperturbative approximations. To actually perform renormalization, or to compute finite quasi-radiative corrections, one will therefore work with a trivially symmetrized DS equation in order to be able to

use normal, symmetric counterterms. (This problem will be mitigated, though not completely removed, by the use of a BS-resummed vertex equation where e.g. the  $(A)_{3'}$  graph of Fig. 4 is replaced by a manifestly Bose-symmetric object, though the replacement for the  $(C)_{3'}$  term still retains an asymmetry).

### 3 Equations for the ghost vertices

For the ghost-gluon-antighost vertex,  $\Gamma_{GV\bar{G}}$ , and its generalized Feynman rule, it is again self-consistent (though not the most general solution) to assume a pure  $f_{abc}$  color structure, the Lorentz structure then being given by

$$\begin{aligned}
[\Gamma_{GV\bar{G}}^{[r,0]}(p, k, -p')]_{abc}^\mu = & if_{abc} [p^\mu \tilde{F}_0^{[r,0]}(p, k, -p') \\
& + k^\mu \tilde{F}_1^{[r,0]}(p, k, -p')] \quad (3.1)
\end{aligned}$$

The dimensionless invariant functions  $\tilde{F}_i$ , with perturbative limits  $\tilde{F}_i^{(0)pert} = \delta_{i0}$ , depend on the invariants  $p^2$ ,  $k^2$ ,  $p'^2$ . Fig. 5 shows the diagrammatic form of the DS equation for  $\Gamma_{GV\bar{G}}$  in its ghost channel, i.e., with the “ $G$ ” leg as the unsymmetrically distinguished leftmost leg. It again features a four-point amplitude  $T'_{GVV\bar{G}}$  which is 1PI in only the “horizontal” channel. Residue-taking both in this equation and in the corresponding equation in the antighost channel again reveals the presence of compensating poles in  $T'$ , and taking these into account one again obtains a rearranged form of the equation as in Fig. 5(b).

When staying strictly in Landau gauge, as we do in this paper, it is actually unnecessary, as far as the self-consistency of the generalized Feynman rule is concerned, to evaluate the terms on the r.h.s. of Fig. 5(b) in detail: brief inspection shows that the latter, at  $\xi = 0$ , do not sustain nonperturbative  $A$  terms. Consider e.g., term  $(B)_G$  of Fig. 5(b) with the momentum assignments shown. Its upper  $\Gamma_{GV\bar{G}}$  vertex has, by (3.1), Lorentz structure

$$(p' - q_2)^\lambda \tilde{F}_0 + q_2^\lambda \tilde{F}_1. \quad (3.2)$$

The internal gluon line carrying momentum  $q_2$  has, in Landau gauge, a transverse projector  $t^{\epsilon\lambda}(q_2)$  so that only the  $p'^\lambda \tilde{F}_0$  portion survives. But  $p'$  is an external momentum not running in the loop, so the integrand loses one power of the loop momentum as compared to standard power counting. Since the loop had only logarithmic divergence to begin with, it is now actually convergent. Analogous arguments apply to the term  $(C)_G$  of Fig. 5(b). The only possible nonperturbative modifications are therefore those from term  $(A)_G$ , if any. These must have at least one denominator factor  $(p^2 + \tilde{u}'_{1,2}\Lambda^2)$  in the momentum variable of the leftmost (ghost) leg. But the same argument applies to the two alternative forms of the ghost-vertex equation not displayed in Fig. 5, with the “antighost” (momentum  $-p'$ ) and gluon (momentum  $k$ ) lines, respectively, as leftmost legs. The only nonperturbative terms in the  $\tilde{F}$  invariant functions of equation (3.1), that are candidates for self-consistency, are therefore those proportional to

$$\frac{\Lambda^6}{(p^2 + \tilde{u}'_{1,2}\Lambda^2)(k^2 + u'_{1,2}\Lambda^2)(p'^2 + \tilde{u}'_{1,2}\Lambda^2)}.$$

These terms, however, make the loop appearing in term  $(A)_G$  of Fig. 5(b) convergent. Since the self-reproduction mechanism of eqs. (1.7/1.8) is dependent upon the  $\frac{1}{\epsilon}$  divergence factor, the amplitude cannot develop any zeroth-order nonperturbative terms in Landau gauge:

$$\begin{aligned} [\Gamma_{GV\bar{G}}^{[r,0]}(p, k, -p')]_{abc}^\mu &= i f_{abc} p^\mu = (\Gamma_{GV\bar{G}}^{(0)pert})_{abc}^\mu \\ (\xi = 0, \text{ all } r). \end{aligned} \quad (3.3)$$

The effect is, of course, basically familiar from perturbation theory: there, the one-loop divergence of the renormalization constant  $\tilde{Z}_1$  vanishes at  $\xi = 0$ . The preceding discussion merely serves as a reminder that in the present context such special divergence reductions also have qualitative dynamical consequences as they suppress the divergence-related self-consistency mechanism. For an amplitude with unphysical degrees of freedom such as  $\Gamma_{GV\bar{G}}$ , it is of course legitimate to depend on the gauge fixing in this way.

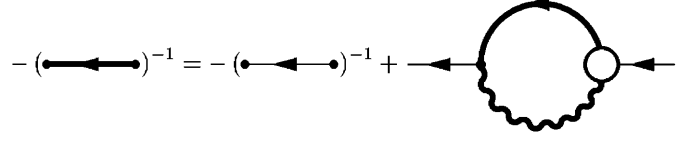
The ghost-self-energy equation, due to its general divergence reduction as discussed in appendix A.2 of [3], also has effectively a logarithmically divergent integral. When evaluated with the purely perturbative vertex (3.3), that integral can produce no more than the *perturbative* divergence, so that again no nonperturbative terms are formed:

$$\tilde{D}^{[r,0]}(p^2) = \tilde{D}^{(0)pert}(p^2) = \frac{1}{p^2} \quad (\xi = 0, \text{ all } r). \quad (3.4)$$

We see that the assumption of [5] that ghost vertices remain perturbative is justified only in Landau gauge.

## 4 Massless-fermion vertices

In the absence of Lagrangian mass terms for quarks, non-perturbative mass scales in the two basic fermion vertices



**Fig. 6.** DS equation for the inverse fermion propagator

$\Gamma_{F\bar{F}}$  and  $\Gamma_{FV\bar{F}}$  can only be multiples of the  $\Lambda$  scale. This case is technically far simpler than the rather complicated situation encountered in the presence of additional RG-invariant scales from “current” quark masses, and is the only one we consider in this paper.

The DS equation for the inverse quark propagator is given diagrammatically in Fig. 6. The corresponding extended Feynman rule at level  $r = 1$ , as discussed in the appendix of [3], is

$$-\Gamma_{F\bar{F}}^{[1,0]}(p) = \not{p} + w_1\Lambda + \frac{w_3\Lambda^2}{\not{p} + w_2\Lambda}. \quad (4.1)$$

Its nonperturbative content is characterized by the three dimensionless, real parameters  $w_1, w_2, w_3$ . Extraction of the self-consistency conditions for these involves calculating the divergent parts of the loop of Fig. 6, evaluated with [1, 0] input elements, and proceeds largely as in the gluon-propagator case considered in [3]. The resulting equations

$$w_2 = w'_2, \quad (4.2)$$

$$\frac{1}{\beta_0} [4w_1 - 4z_1] = w_1, \quad (4.3)$$

$$\frac{1}{\beta_0} [4w_1 z_1 - 4z_2] = w_3, \quad (4.4)$$

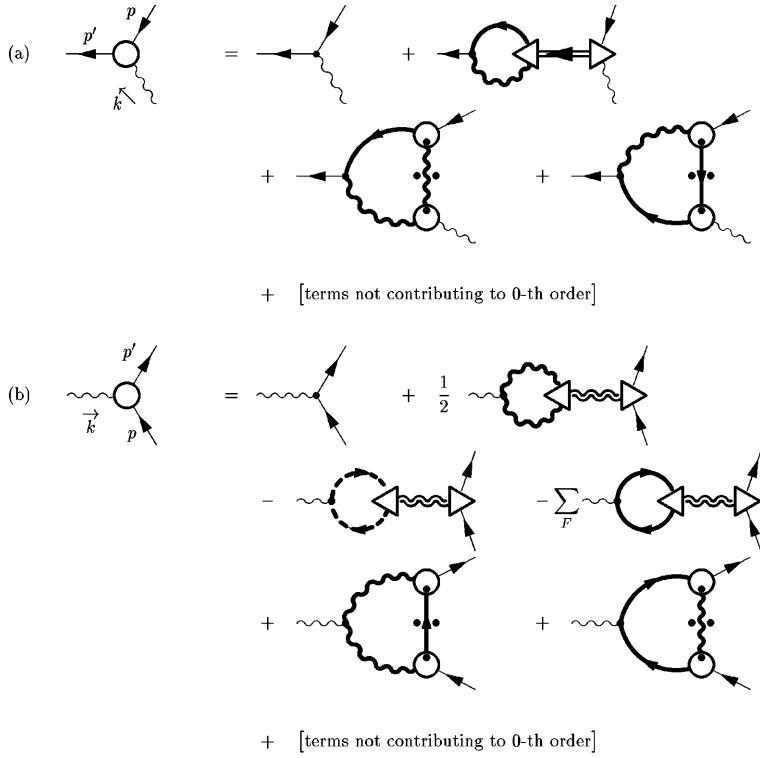
make reference to the parameters  $z$  of the quark-gluon three-point vertex,  $\Gamma_{FV\bar{F}}$ . Its transverse-gluon projection at level  $r = 1$  reads,

$$\begin{aligned} [\Gamma_{FV\bar{F}}^{[1,0]}(-p', k, p)]^\nu &= \\ t^{\nu\mu}(k) &\left\{ \gamma^\mu + z_1 \left( \frac{\Lambda}{\not{p}' + w'_2\Lambda} \gamma^\mu + \gamma^\mu \frac{\Lambda}{\not{p} + w'_2\Lambda} \right) \right. \\ &+ z_2 \frac{\Lambda}{\not{p}' + w'_2\Lambda} \gamma^\mu \frac{\Lambda}{\not{p} + w'_2\Lambda} \\ &+ \frac{\Lambda^2}{k^2 + \tilde{u}'_2\Lambda^2} \left[ z_3 \gamma^\mu + z_4 \left( \frac{\Lambda}{\not{p}' + w'_2\Lambda} \gamma^\mu + \gamma^\mu \frac{\Lambda}{\not{p} + w'_2\Lambda} \right) \right. \\ &\left. \left. + z_5 \frac{\Lambda}{\not{p}' + w'_2\Lambda} \gamma^\mu \frac{\Lambda}{\not{p} + w'_2\Lambda} \right] \right\}. \end{aligned} \quad (4.5)$$

Here the notation of [3] for the dimensionless coefficients has been changed and simplified somewhat, the relation [3]  $\rightarrow$  this paper being given by

$$\begin{aligned} z_{0,1}^{[1]} &\rightarrow z_1, & z_{0,4}^{[1]} &\rightarrow z_2, & z_{1,0}^{[1]} &\rightarrow z_3, \\ z_{1,1}^{[1]} &\rightarrow z_4, & z_{1,4}^{[1]} &\rightarrow z_5. \end{aligned} \quad (4.6)$$

We have from the outset omitted all terms that would lead to conflict with perturbative divergence degrees. For



**Fig. 7.** Equivalent DS equations for the quark-antiquark-gluon vertex in (a) fermionic channel and (b) gluonic channel

$\Gamma_{FV\bar{F}}$  there are two DS equations, one in the “fermionic” and one in the “gluonic” channel, which are equivalent for the exact vertex but in general will give rise to different approximations. Rearranged for one-quark-shadow irreducibility in the now familiar way, these are depicted in Figs. 7(a) and 7(b), respectively. While each of the two forms, due to the compensating-poles mechanism, suffers from “loss of equations” in the sense discussed in Sect. 2.3, it is interesting that *the two forms taken together* produce just the required number of self-consistency conditions [9]:

$$\frac{1}{\beta_0} \left[ \frac{9}{4} z_1 - \frac{9}{4} \frac{1}{w_3} z_1 z_2 - \frac{1}{u_3} \left( \frac{15}{2} x_1 - \frac{5}{4} x_3 \right) z_4 + \frac{2}{3} N_F \frac{1}{u_3} z_3 z_4 \right] = z_1 \quad (4.7)$$

$$\frac{1}{\beta_0} \left[ \frac{9}{4} z_1^2 - \frac{9}{4} \frac{1}{w_3} z_2^2 - \frac{1}{u_3} \left( \frac{15}{2} x_1 - \frac{5}{4} x_3 \right) z_5 + \frac{2}{3} N_F \frac{1}{u_3} z_3 z_5 \right] = z_2 \quad (4.8)$$

$$\frac{1}{\beta_0} \left[ \frac{9}{4} z_1 - \frac{9}{4} \frac{1}{u_3} x_1 z_4 \right] = z_1 \quad (4.9)$$

$$\frac{1}{\beta_0} \left[ \frac{9}{4} x_3 - \frac{9}{4} \frac{1}{u_3} x_4 z_3 \right] = z_3 \quad (4.10)$$

$$\frac{1}{\beta_0} \left[ \frac{9}{4} x_3 z_1 - \frac{9}{4} \frac{1}{u_3} x_4 z_4 \right] = z_4 \quad (4.11)$$

In addition, the quasi-perturbative remainders contain divergences given by

$$\left( \Xi_{FT\bar{F}}^{(1)} \right)^\mu = \gamma^\mu \left[ \frac{9}{4} - \frac{9}{4} \frac{1}{w_3} z_1^2 - \frac{1}{u_3} \left( \frac{15}{2} x_1 - \frac{5}{4} x_3 \right) z_3 + \frac{2}{3} N_F \frac{1}{u_3} z_3^2 \right], \quad (4.12)$$

$$\left( \Xi_{F\bar{F}}^{(1)} \right)^\mu = \gamma^\mu \left[ \frac{9}{4} - \frac{9}{4} \frac{1}{u_3} x_1 z_3 \right], \quad (4.13)$$

for Figs. 7(a) and 7(b) respectively, which differ from the perturbative quantity

$$z_{FV\bar{F}}^{(1)}(\xi=0) = \frac{9}{4} \quad (4.14)$$

by defect terms involving the vertex constants  $x$  and  $z$ , which again cannot be forced to zero at  $r=1$  but only asymptotically at large  $r$ .

Note that there is no equation determining the vertex-pole position  $w'_2$ , which by (4.2) is also the propagator-zero position.

## 5 Solution for 2- and 3-point coefficients

### 5.1 Analysis of equations

The system of self-consistency conditions for the  $r=1$  nonperturbative coefficients, as established up to now, consists of eqs. (4.8/4.30/4.31) of [3] for the gluonic self-energy parameters  $u_1, u_2, u_3$ , which now assume<sup>1</sup> the sim-

<sup>1</sup> Eq. (5.2) corrects for a misprint in (4.31) of the first reference of [3], where a tadpole contribution  $-9N_C u_{1,1}/4$  appears with the wrong sign

pler forms

$$u'_2 = \bar{u}'_2 = u_2, \quad (5.1)$$

$$\frac{1}{\beta_0} \left[ \frac{9}{4} u_1 - \frac{33}{2} x_1 + \frac{5}{4} x_3 - 2N_F \left( w_3 + (w_1 + w_2) z_1 + z_2 - \frac{1}{3} z_3 \right) \right] = u_1, \quad (5.2)$$

$$\frac{1}{\beta_0} \left[ \frac{5}{2} u_2 \left( 3x_1 - \frac{1}{2} x_3 \right) + 9u_1 x_3 - 9x_4 - 2N_F \left( z_3 \left( w_3 + \frac{1}{2} u_2 \right) + (w_2 - w_1) z_4 + z_5 \right) \right] = u_3, \quad (5.3)$$

plus eqs. (2.43-2.47) above for the 3-gluon-coefficients  $x_1 \dots x_5$ , plus eqs. (4.2-4.4) and (4.7-4.11) above for the self-energy coefficients  $w_1 \dots w_3$  and vertex coefficients  $z_1 \dots z_5$  of massless fermions. (In (5.1),  $\bar{u}'_2$  is the gluonic denominator parameter from the fermion vertex, defined in (4.5) above). Its peculiar properties, in particular with respect to under- and overdetermination tendencies, can be summarized as follows.

(a) One observes that the system as a whole exhibits a *scaling property*: any one of the coefficients that is presumed to be nonzero may be divided out of these equations, while replacing the others by their *ratios* to this one or to a uniquely fixed power of it, and rescaling  $\Lambda$  accordingly. This property, which in a nonlinear system is non-trivial, is a natural consequence of the scheme-blindness of the basic self-consistency mechanism: a rescaling of  $\Lambda$ , which corresponds to a change of scheme, will not change the form of the zeroth-order conditions. We will choose, for definiteness, a rescaling by the 3-gluon coefficient  $x_1$  of (2.11):

$$\tilde{\Lambda}^2 = x_1 \Lambda^2, \quad (5.4)$$

$$\begin{aligned} \tilde{u}_1 &= \frac{u_1}{x_1}, \quad \tilde{u}_2 = \frac{u_2}{x_1}, \quad \tilde{u}_3 = \frac{u_3}{x_1^2}, \\ \tilde{x}_1 &= 1, \quad \tilde{x}_2 = \frac{x_2}{x_1^2}, \quad \tilde{x}_3 = \frac{x_3}{x_1}, \quad \tilde{x}_4 = \frac{x_4}{x_1^2}, \quad \tilde{x}_5 = \frac{x_5}{x_1^3}, \\ \tilde{Z}_1 &= \frac{Z_1}{x_1}, \quad \tilde{Z}_2 = \frac{Z_2}{x_1^2}, \\ \tilde{w}_1 &= \frac{w_1}{\sqrt{x_1}}, \quad \tilde{w}_2 = \frac{w_2}{\sqrt{x_1}}, \quad \tilde{w}_3 = \frac{w_3}{x_1}, \\ \tilde{z}_1 &= \frac{z_1}{\sqrt{x_1}}, \quad \tilde{z}_2 = \frac{z_2}{x_1}, \quad \tilde{z}_3 = \frac{z_3}{x_1}, \quad \tilde{z}_4 = \frac{z_4}{\sqrt{x_1^3}}, \quad \tilde{z}_5 = \frac{z_5}{x_1^2}. \end{aligned} \quad (5.5)$$

The reason for this choice is that by putting  $x_1 = 0$  one would end up with only the trivial solution (all nonperturbative coefficients vanishing), so one is not losing interesting solutions by assuming  $x_1 \neq 0$ . (For the scaling of the fermionic parameters, we are for the moment assuming  $x_1$  to be positive). In effect, the rescaling *reduces the number*

*of unknowns by one* while introducing the modified  $\tilde{\Lambda}$  of (5.4), scaled by an unknown factor.

The rescaled system of fourteen conditions, with fifteen unknowns and  $\tilde{Z}_1, \tilde{Z}_2$  as external parameters, has solutions coming in pairs: one checks that if

$$\left\{ \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{x}_2 \dots \tilde{x}_5, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4, \tilde{z}_5 \right\} \quad (5.6)$$

is a solution, then for the same  $\tilde{Z}_1$  and  $\tilde{Z}_2$  the set

$$\left\{ \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, -\tilde{w}_1, -\tilde{w}_2, \tilde{w}_3, \tilde{x}_2 \dots \tilde{x}_5, -\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, -\tilde{z}_4, \tilde{z}_5 \right\} \quad (5.7)$$

is also a solution, which we shall refer to as a “mirror” solution. Note that this discrete ambiguity affects only fermionic parameters.

(b) We have not obtained equations fixing the vertex-denominator parameters ( $u_2$  and  $w_2$  in the present case). Neither analysis of the 4-gluon vertex [4] nor, as preliminary studies indicate, use of “resummed” DS equations will change this situation. We did obtain conditions like the  $w'_2 = w_2$  of (4.2), and the corresponding  $u'_2 = \bar{u}'_2 = u_2$  of (5.1), which ensure one common pole position in all basic vertices for a given type of external leg (and also the presence of propagator zeroes at the positions of vertex poles), but  $u_2$  and  $w_2$ , in the end, have no determining equations of their own. This leads to the unexpected conclusion that the divergent parts of the momentum-space DS equations as used up to now do not yet determine the nonperturbative  $\Lambda$  dependence completely. The reason is that these equations, in a sense, do not provide enough divergence. Indeed, the quadratically divergent gluon self-energy is the only vertex having  $u_2$  and  $w_2$  appear at least on the *right-hand* sides of its self-consistency conditions for  $u_1$  and  $u_3$ , but at least two more equations with the same or higher degree of divergence would be needed to “lift” the two parameters from the denominators of loop integrands into numerator expressions that provide self-consistency conditions – a feat that only divergent integrations can perform.

Additional conditions for fixing  $u_2$  and  $w_2$  therefore should have general compatibility with the momentum-space DS equations and provide sufficient divergence. The only natural candidates here are those requiring the vanishing of the “equation-of-motion condensates”, i.e. of vacuum expectations of the simplest (dimension four) local composite operators proportional to the left-hand sides of the field equations. We use the condensate conditions for the ghost and fermion fields in the form

$$(g_0 \nu_0^\epsilon)^2 \langle 0 | \bar{c}_a(x) \{ [\delta_{ab} \square + g_0 \nu_0^\epsilon f_{abc} A_c^\mu(x) \partial^\mu] \times c_b(x) \} | 0 \rangle = 0 \quad (5.8)$$

$$(g_0 \nu_0^\epsilon)^2 \langle 0 | \bar{\psi}(x) \{ [i \not{\partial} + g_0 \nu_0^\epsilon \not{A}(x)] \psi(x) \} | 0 \rangle = 0 \quad (5.9)$$

In momentum space these are, of course, nothing but the ghost and quark propagator equations integrated over

momentum space, or equivalently, taken at zero separation in coordinate space. They are therefore obviously compatible with, and natural completions of, the unintegrated (momentum space) or nonzero-separation (coordinate space) DS equations we have exploited up to now. In standard integral-equation theory with convergent integrals and well-behaved functions, they would not represent independent statements, but in a theory with divergent loop integrals and therefore in need of renormalization, they do carry new information: since they involve operators of higher compositeness, they possess new divergences leading to new *zeroth-order* conditions on the nonperturbative coefficients. At the  $r = 1$  level these conditions read,

$$\frac{3}{\beta_0^2} [-u_1^2 + u_3] \Lambda^4 = 0 \quad (5.10)$$

$$\left\{ \begin{aligned} & \frac{3}{\beta_0} [w_1^4 - 4w_1^2 w_3 - 2w_1 w_2 w_3 - w_2^2 w_3 + w_3^2] \\ & + \frac{2}{\beta_0^2} [u_3 - u_1^2 - 3u_1 w_3 - 6w_1^4 + 12w_1^2 w_3 + 6w_1 w_2 w_3 \\ & + 3z_1(u_1 w_1 - u_1 w_2 + 4w_1^3 + 2w_1^2 w_2 + 2w_1 w_2^2 \\ & \quad - 6w_1 w_3 - 2w_2 w_3) \\ & + 3z_2(-u_1 - 2w_1^2 - 2w_1 w_2 - 2w_2^2 + 2w_3) \\ & + z_3(u_1 + u_2 + 3w_3) + 3z_4(w_2 - w_1) + 3z_5] \end{aligned} \right\} \Lambda^4 = 0 \quad (5.11)$$

Condition (5.10), from the ghost equation of motion, notably provides a restriction on *gluonic* parameters (it is, incidentally, equivalent to requiring the vanishing of the zeroth-order, dimension-two gluon condensate  $\langle A^\mu A^\mu \rangle$ , which at  $r = 1$  turns out to be proportional to  $-u_1^2 + u_3$ ). Since in Landau gauge at one loop it is the only condition from the ghost sector, is independent of the presence of fermions, and of remarkable simplicity, we give it priority in complementing eqs.(5.2/5.3). Condition (5.11) brings in, in addition, the fermionic parameters, and is suitable for complementing eqs. (4.3/4.4). The order-[1,0] equation-of-motion condensate for the gluon field,

$$\begin{aligned} & (g_0 \nu_0^\epsilon)^2 \langle 0 | \left[ \frac{1}{2} (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)^2 \right. \\ & + \frac{3}{2} (g_0 \nu_0^\epsilon) f_{abc} (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) A_b^\mu A_c^\nu \\ & + (g_0 \nu_0^\epsilon)^2 f_{abe} f_{cde} A_a^\mu A_b^\nu A_c^\mu A_d^\nu + (g_0 \nu_0^\epsilon) f_{abc} (\partial^\mu \bar{c}_a) A_c^\mu c_b \\ & \left. + (g_0 \nu_0^\epsilon) N_F \bar{\psi} \not{A} \psi \right] | 0 \rangle = 0, \end{aligned} \quad (5.12)$$

is the most complicated, and its zeroth-order form at  $r=1$ ,

$$\left\{ \begin{aligned} & \frac{24}{\beta_0} [u_1^2 - u_3] \\ & + \frac{1}{\beta_0^2} \left[ -390u_1^2 + 228u_3 - 324x_4 + 594u_1 x_1 \right. \\ & \quad \left. + 279u_1 x_3 + 270u_2 x_1 - 45u_2 x_3 \right] \\ & + \frac{N_F}{\beta_0^2} \left[ 8u_1^2 - 8u_3 + 24u_1 w_3 + 48w_1^2 \right. \\ & \quad - 96w_1^2 w_3 - 48w_1 w_2 w_3 \\ & \quad + 24z_1(-u_1 w_1 + u_1 w_2 - 4w_1^3 - 2w_1^2 w_2 \\ & \quad - 2w_1 w_2^2 + 6w_1 w_3 + 2w_2 w_3) \\ & \quad + 24z_2(u_1 + 2w_1^2 + 2w_1 w_2 + 2w_2^2 - 2w_3) \\ & \quad \left. - 8z_3(u_1 + u_2 + 3w_3) + 24z_4(w_1 - w_2) - 24z_5 \right] \\ & + \frac{1}{\beta_0^3} \left[ \text{3-loop-terms} \right] \end{aligned} \right\} \Lambda^4 = 0, \quad (5.13)$$

involves the largest number of vertex parameters simultaneously. In the present context it cannot be applied in its exact form; the 3-loop-terms must be omitted for formal consistency with the omission of 2-loop-terms in the  $l = 1$  gluon-self-energy calculation. For these reasons, condition (5.13) may be expected to be the most difficult to fulfill on our level of approximation, and is not one of our primary choices for completing the self-consistency system. We will check in the end to what extent it can be accommodated.

(c) Due to the large dimensions and considerable overdetermination of the 4-gluon self-consistency problem to be discussed in [4], the total coupled problem cannot, in our experience, be attacked directly with currently existent mathematical software tools. However, we have already emphasized the (also unexpected) result that the 4-gluon-vertex problem couples to the fewer-point amplitudes only through the narrow “bottleneck” of two 4-gluon-coefficient combinations (2.49/2.50) appearing in only three of the 3-gluon conditions. This situation of a *near decoupling of the 4-gluon self-consistency problem* renders the following strategy sensible (it is, in any case, the only practical strategy at present). One omits, as a first step, the 4-gluon conditions completely, and treats the parameters  $Z_1, Z_2$  appearing in (2.45-2.47) as two additional unknowns in the 2-point-plus-3-point-system, which thereby becomes doubly underdetermined. Combined with the scaling property of point (a), which effectively reduces the number of unknowns by one, this results in an *effective one-parameter freedom* in the solution of the 2-plus-3-point problem. Since the number of calculable coefficients – in the present case, fourteen in the 2-and-3-point amplitudes, and seventeen in the “minimal” four-gluon vertex to be discussed in [4] – is much larger, the solutions will still be nontrivial and informative; in particular, one may explore in what range, if any, of this one-parameter freedom there exist physically acceptable solutions.

In a second step, which we defer to [4], one may then adjoin the values of  $Z_1, Z_2$  thus determined as additional constraints to the 4-gluon self-consistency problem: this will represent only a minor increase in the anyway massive



overdetermination of that problem. Since the 4-gluon system refers to the 2-and-3-point coefficients, it inherits the effective one-parameter freedom, and to within that freedom may be dealt with separately, with methods adapted to its overdetermined nature.

It would seem that any one coefficient or combination of coefficients of the 2-and-3-point system could be used to parametrize the effective one-parameter freedom; in particular, some combination of the quantities (2.49/2.50), which caused the freedom in the first place, would seem to be a natural parameter. However, one again faces unexpected restrictions here: due to the peculiar structure of the system in its fermionic unknowns, the fixing of a combination of non-propagator parameters, instead of rendering the system well-determined, usually splits it into an over- and an underdetermined part. The parametrizing quantity should thus refer to propagator coefficients. In the following we will choose, for no other reasons than technical simplicity, the rescaled quark self-energy coefficient  $\tilde{w}_1$ .

(d) At present, the question remains open as to whether there exists a preferred or natural way of finally removing the one-parameter freedom. One might think of recalculating the quantities  $\tilde{Z}_1$ ,  $\tilde{Z}_2$  later from the least-squares four-gluon solution and see if there is a parameter range where they agree, at least qualitatively, with those from the 2-and-3-point solution. We shall indeed do this in [4], but shall see that in the parameter range where the entire solution is physically acceptable, a mismatch is unavoidable at  $r = 1$ , although small in the case of a pure gluon theory. Alternatively, the vanishing of any of the previously noted approximation errors existing at the ( $r = 1$ ,  $l = 1$ ) level could be used as a condition. The common problem of all conditions of this kind is that (i) there are several of them, and any selection from among them appears arbitrary, (ii) they are mostly so restrictive that their imposition leaves only the trivial solution, with all nonperturbative coefficients vanishing. The message the defect terms seem to convey is that the still rather simple and rigid structure of the  $r = 1$  system of approximants entails unavoidable approximation errors that cannot be forced to zero without overstraining that structure; they can disappear only gradually as  $r$  is increased.

Our (arguable) guideline, then, in choosing conditions for fixing the zeroth-order nonperturbative coefficients, is to give priority to the primary DS self-consistency conditions in the wider sense (including the momentum-integrated conditions) and to accommodate as many of these as possible with physically acceptable solutions, but to live with approximation errors in all other secondary conditions, such as desirable structures of order- $g^2$  terms, differential ST relations, etc.

## 5.2 Discussion of solutions

The system augmented by (5.10) at a fixed value of  $\tilde{w}_1$  may be reduced by successive elimination to an algebraic equation of the 10th degree for the quantity  $\tilde{w}_3$ . The other

coefficients can then be calculated recursively from the solutions of this equation and (5.11), and depend parametrically on  $\tilde{w}_1$ . (These calculations have been performed using the MAPLE V computer-algebra system). The following noteworthy features emerge.

(a) When assuming  $x_1 < 0$  and performing the rescaling (5.5/ 5.4) with  $|x_1| = -x_1$  instead of  $x_1$ , the ten roots obtained for  $\tilde{w}_3$  are all complex. Such solutions can immediately be discarded as unphysical, since they lead to vertex functions not real at real Euclidean momenta, and the nonlinear nature of the system permits no superposition to obtain real solutions. Therefore no physical solutions have been lost by assuming  $x_1 > 0$  and rescaling as in (5.5/ 5.4).

(b) Over a range  $0.3 \leq \tilde{w}_1 \leq 1.2$  (all ranges quoted are approximate) only eight of the ten  $\tilde{w}_3$  roots come in complex-conjugate pairs, but two are real: *there exist solutions with all vertex coefficients real*. This result is entirely nontrivial, and represents substantial progress over the earlier attempt of [5], where what we would now call the  $r = 1$  level of approximation was studied in a more heuristic fashion, with strong a priori simplifications of the vertex approximants, and without taking the compensating-poles mechanism into account. There, only partly real solutions could be found.

Of the two real  $\tilde{w}_3$  roots, one is negative and one positive. The negative  $\tilde{w}_3$  value always turns out to lead to ‘‘tachyonic’’ pole positions (negative values of the  $\rho_{\pm}^2$  of (5.18) below) in at least one of the two propagators, and can also be discarded as unphysical. Again it is nontrivial and noteworthy that only one of several solutions of the nonlinear system stands out as a candidate for a physical solution. Each of the two real solutions still exhibits the doubling of eqs. (5.6/5.7), i.e. has a mirror solution for some of its fermionic parameters in the range  $-1.2 \leq \tilde{w}_1 \leq -0.3$ .

(c) For the solution in the range  $0.3 \leq \tilde{w}_1 \leq 1.2$  with  $\tilde{w}_3$  real and positive, Table 1 records the nature of the poles in the Euclidean transverse-gluon and fermion propagators, which now read

$$D_T^{[1,0]}(k^2) = \frac{k^2 + u_2 \Lambda^2}{(k^2 + \sigma_+ \Lambda^2)(k^2 + \sigma_- \Lambda^2)}, \quad (5.14)$$

$$S^{[1,0]}(\not{p}) = \frac{\not{p} + w_2 \Lambda}{(\not{p} + \rho_+ \Lambda)(\not{p} + \rho_- \Lambda)}, \quad (5.15)$$

(see eqs. (2.6) and (4.1)). One finds that over a narrower range of  $\tilde{w}_1$ , namely,

$$0.5 \leq \tilde{w}_1 \leq 0.9 \quad (5.16)$$

the solution with the real and positive  $\tilde{w}_3$  root in addition fulfills the three inequalities

$$\begin{aligned} \gamma_V^2 &\equiv u_3 - \left(\frac{u_1 - u_2}{2}\right)^2 > 0, \\ \gamma_F^2 &\equiv w_3 - \left(\frac{w_1 - w_2}{2}\right)^2 > 0, \quad |w_1 + w_2| > 0, \end{aligned} \quad (5.17)$$

so that *both propagators simultaneously* exhibit complex-conjugate pole pairs at  $k^2 = -\sigma_{\pm} \Lambda^2$  and  $p^2 = -\rho_{\pm}^2 \Lambda^2$ ,

**Table 1.** Ranges of interest for propagator-pole parameters

	$0.3 < \tilde{w}_1 < 0.4$	$0.4 < \tilde{w}_1 < 0.5$	$0.5 < \tilde{w}_1 < 0.9$	$0.9 < \tilde{w}_1 < 1.1$	$1.1 < \tilde{w}_1 < 1.2$
$\tilde{\sigma}_+$	$< 0$		complex	$< 0$	$< 0$
$\tilde{\sigma}_-$	$> 0$		conjugate	$< 0$	$< 0$
$\tilde{\rho}_+^2$	$> 0$	$> 0$		complex	$> 0$
$\tilde{\rho}_-^2$	$> 0$	$> 0$		conjugate	$> 0$

**Table 2.** Typical solution with  $N_F = 2$ 

$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\gamma_V$
-0.3604	-0.4884	+0.1299	+1.0000	-8.7433	+8.9088	-3.2607	-6.2711	0.3547
$\tilde{w}_1$	$\tilde{w}_2$	$\tilde{w}_3$	$\tilde{z}_1$	$\tilde{z}_2$	$\tilde{z}_3$	$\tilde{z}_4$	$\tilde{z}_5$	$\gamma_F$
+0.6749	+0.6749	+0.1202	-0.9561	-0.9356	-0.4282	+0.4094	+0.2242	0.3468

where

$$\begin{aligned}\sigma_{\pm} &= \frac{u_1 + u_2}{2} \pm i\gamma_V, \\ \rho_{\pm}^2 &= \frac{w_1^2 + w_2^2}{2} - w_3 \pm i(w_1 + w_2)\gamma_F.\end{aligned}\quad (5.18)$$

Thus there exist solutions in which the elementary excitations of the two basic QCD fields are both short-lived. In the present framework this is the essential indicator of confinement, since it implies the vanishing of S-matrix elements with external single-gluon or single-quark legs [5]. We again regard it as nontrivial that a parameter range should at all exist in which this situation prevails. Note also that in the gluonic portion of Table 1, there is always at least one ‘‘tachyonic’’ gluon-propagator pole outside the slightly wider range  $0.4 \leq \tilde{w}_1 \leq 0.9$ ; we view it as significant that the only solutions with real vertices and non-tachyonic gluons have gluon propagators with complex pole pairs. – Over the range (5.16), most of the other vertex coefficients are only weakly varying.

This interesting solution still has a ‘‘mirror’’ solution in the sense of (5.7), i. e. in a  $\tilde{w}_1$  range which is the negative of (5.16). We are not aware of a *theoretical* criterion that would resolve this discrete ambiguity. It appears to be the remnant of the  $U(1)_{L-R}$  invariance of the massless Lagrangian under different rephasings of the left-handed and right-handed quark fields, which gets broken down to a  $Z_2$  (with elements  $\phi_L - \phi_R = 0$  or  $\pi$ ) by the dynamical generation of chirally noninvariant terms in our fermion functions, the  $\phi_L - \phi_R = \pi$  giving an invariance of the self-consistency equations when combined with  $\Lambda \rightarrow -\Lambda$ . This ambiguity is shared by quantities like the  $r = 1$ , zeroth-order fermion condensate

$$\left(g_0^2 \langle 0 | \bar{\Psi} \Psi | 0 \rangle\right)^{[1,0]} = \frac{12}{\beta_0} (w_1^3 - 2w_1w_3 - 2w_2w_3) \Lambda^3, \quad (5.19)$$

as a prototype chirally noninvariant matrix element. In quantities which are in principle observable, only combinations insensitive to this mirror ambiguity would be expected to occur.

(d) Upon imposing condition (5.13), from the gluonic equation of motion, to remove the residual freedom in the  $\tilde{w}_1$ , we find values  $\tilde{w}_1$  not only outside the range (5.16) but in fact outside the larger range of Table 1, where  $\tilde{w}_3$  and thus the entire solution ceases to be real and physically acceptable: enforcing (5.13) one loses the possibility of a physical solution and of all the features noted in (c). Thus (5.13), like removal of the order- $g^2$  defect terms noted above, seems to be a strongly restrictive condition that the simple  $r = 1$  structure is too rigid to accommodate.

Within the general strategy suggested and used here, it appears that restriction to the quite limited parameter range where all propagator singularities are complex conjugate, and none tachyonic, in itself represents a sensible limitation to the one-parameter freedom, and one that is difficult to narrow further without overburdening the  $r = 1$  approximation.

(e) For use of the generalized Feynman rules in applications, we list in Table 2 a typical set of two-point and three-point vertex coefficients for  $N_F = 2$ , for  $\tilde{w}_1$  chosen in about the middle of the range (5.16). (Since in that range  $\tilde{w}_2$  varies slowly and is itself of modulus  $\approx 0.7$ , we choose the point  $\tilde{w}_1 = \tilde{w}_2$  for simplicity.) This set still needs completion through the corresponding four-gluon vertex coefficients  $\zeta_i$ , but since many lower-order calculations need at most 3-point vertices, it seems legitimate to defer presentation of these to [4]. The propagator-pole parameters (5.18) for this solution are

$$\begin{aligned}\sigma_{\pm} &= (-0.4245 \pm i0.3547)\tilde{\Lambda}^2, \\ \rho_{\pm}^2 &= (0.3353 \pm i0.4679)\tilde{\Lambda}^2.\end{aligned}\quad (5.20)$$

For purposes of comparison, we also briefly look at solutions for the pure-gluon theory ( $N_F = 0$ ). Here the parametrizing quantity may be taken to be the gluonic vertex coefficient  $\tilde{x}_3$ , and again we choose a typical value,  $\tilde{x}_3 \approx 1$ , from the (again existing) range in which the gluonic propagator poles are complex conjugate. (The value  $\tilde{x}_3 = 1$ , incidentally, is also one which symmetrizes, though not removes, the defects of (2.52) in the perturbative three-gluon divergence). Table 3 lists coefficients for this

**Table 3.** Typical solution for pure-gluon system

$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\gamma_V$
-1.7429	+0.8456	+3.0376	+1.0000	-6.1825	+1.0000	-4.8682	+28.605	1.1650

case. It is still impossible here to accommodate condition (5.13) in a physically acceptable solution. Note however that this solution is now unaffected by the doubling of (5.7).

The identity (2.10) as applied to the  $[r, 0]$  functions (2.5) and (2.11) would imply the relations

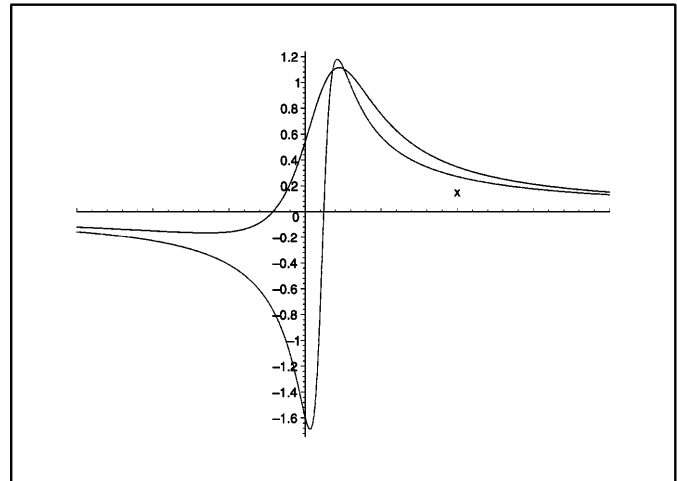
$$\frac{\tilde{x}_3}{\tilde{u}_2} = 0, \quad \frac{\tilde{x}_4}{\tilde{u}_2} + \tilde{x}_1 = 0, \quad \frac{\tilde{x}_5}{\tilde{u}_2} + \tilde{x}_2 = -\tilde{u}_3. \quad (5.21)$$

In the typical solutions of Tables 2 and 3, all three relations have large errors, in the sense that the violations are of roughly the same magnitude as typical  $x$  coefficients themselves. The same is true for other approximation errors such as the perturbative-limit deviations in the one-loop divergence of (2.52); one sees that the  $r = 1$  approximants with their relatively small numbers of coefficients are still far from being able to satisfy all the additional restrictions one would like to implement.

For the system with  $N_F = 2$ , there are several indications that the approximation substantially exaggerates the effects of the massless-fermion loops on the zeroth-order vertices. In particular, for the three-gluon vertex coefficients, almost no trace remains of the pattern of signs and orders of magnitude established by the pure-gluon system – a much stronger effect than one would expect from, say, the  $N_F$  contributions to one-loop renormalization-group functions. Another aspect emerges in Fig. 8 where the transverse-gluon propagator function  $D_T(k^2)$  (in units of  $\tilde{\Lambda}^{-2}$ ) is plotted for the typical solutions of Tables 2 and 3. The value at  $k^2 = 0$  is in general a finite constant whose sign is determined by the  $u_2$  parameter; it is positive for the pure-gluon system. The quark-antiquark self-energy contributions with their negative signs generally tend to lower the value of  $D_T(0)$ , but Fig. 8 shows that their effect here is greatly overestimated so that they pull the value down to zero “too early”, i.e. somewhat into the Euclidean domain, resulting in a violation of reflection positivity in a small neighborhood of  $k^2 = 0$ . In [4] we will encounter more indications that the solution with fermions at  $r = 1$  seems to meet extra difficulties not yet present for the pure Yang-Mills case.

## 6 Conclusion

We have demonstrated the feasibility of a self-consistent determination of generalized Feynman rules, accounting for the nonperturbative  $\Lambda$  dependence of correlation functions through a modified iterative solution, at the simplest level of systematic approximation of that dependence. We have shown, and regard it as nontrivial, that the nonlinear self-consistency problem admits physically acceptable



**Fig. 8.** Transverse gluon propagator  $\tilde{\Lambda}^2 \cdot D_T(x)$  as function of dimensionless variable  $x = k^2/\tilde{\Lambda}^2$  for  $N_F = 0$  (upper curve) and  $N_F = 2$  (lower curve). Abscissa unit is  $\Delta x = 0.4$ , abscissa range  $x = -6.0 \dots + 8.0$

solutions, that these stand out clearly against a majority of unphysical ones, and that there exist solutions in which both of the elementary excitations of the basic QCD fields exhibit the short-ranged propagation described by complex-conjugate propagator poles.

It is useful to recall the restrictions under which we have studied this self-consistency problem. We have considered the  $r = 1$  level of rational approximation of the  $\Lambda$  dependence – the lowest level of interest for a “confining” theory like QCD. The limitations inherent in this low approximation order have become clearly visible; its structure is far too simple and rigid to satisfy all desirable conditions and restrictions simultaneously. In particular, we have seen indications that the  $r = 1$  approximation for the system with massless fermions ( $N_F = 2$ ) fares worse than for the pure-gluon system, as it exaggerates the effects of closed fermion loops.

We have worked with the “ordinary” DS equations only, with bare vertices on their distinguished, left-hand external legs. We have evaluated the self-consistency conditions on the one-loop level, in Landau gauge, and with a special decoupling of the 4-gluon-conditions as suggested by the peculiar “bottleneck” structure of the system. It is desirable for future work to gradually remove these limitations, and in particular to study the Bethe-Salpeter-resummed forms of the vertex equations, which may have more of the important physical effects shifted into the low loop orders.

Even with such improvements, two problems are certain to persist that have emerged clearly from the present study. One, which arises only when studying vertices with

at least three legs, and has nothing to do with the specifics of the present method, is the overdetermination dilemma unavoidable when seeking approximate-but-symmetric solutions to the not manifestly symmetric DS equations. This may be “swept under the rug” by trivial symmetrizations, but only at the expense of depriving oneself of an important measure of error. Note that we have nevertheless managed to keep our *zeroth-order* three-point vertices symmetric, the problem being one of order- $g^2$  terms.

The second problem is that the DS equations, through their divergent parts, do not fix the common set of denominator parameters of the approximants. A way of understanding this interesting result is to recall the relation with the operator-product expansion, as discussed in Sect. (2.3) of [3]: the OPE, in its higher orders, contains vacuum expectations of local operators of arbitrarily high compositeness, whereas the DS equations contain at most insertions of three operators at the same spacetime point. The extra composite-operator renormalizations required by those higher condensates represent extra information which the usual DS system does not supply directly. Equation-of-motion condensate conditions, which do represent statements about quantities of higher compositeness, are capable of supplying the extra information, and natural complements insofar as they are special, zero-separation cases of DS equations. Their role, which in the present context may still look marginal, will clearly become more central when going to higher levels  $r$ .

It should be kept in mind that for a confining field system such as QCD, the generalized Feynman rules as considered here allow only the calculation of off-shell Green’s functions of the elementary fields. These still carry little observable information, although the spectrum of the elementary excitations as determined by the singularities of the two-point-functions does constitute important *qualitative* information. To calculate on-shell amplitudes, whose external legs are bound states, one would need in addition bound-state vertices to sit at the outer corners of S-matrix diagrams. These have not been touched upon in this paper, since they are conceptually quite different from the zeroth-perturbative-order quantities: they arise from partial (ladder or improved-ladder) resummation of quasi-perturbative corrections  $g^{2p} I_N^{(r,p)}$ , with  $p \geq 1$ , for

certain superficially *convergent* amplitudes  $I_N$ , in which the mechanism of (1.8) plays no role. Their determination must therefore rely on the established Bethe-Salpeter methods for bound states.

On the other hand we do believe that the calculations described here achieve something new by dealing with the complete set of superficially divergent QCD vertices in one consistent approximation, and that they demonstrate a nontrivial, renormalization-related way of how the renormalization-group invariant mass scale establishes itself in the correlation functions of an asymptotically free theory.

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